

RICCI FLOWS ON SURFACES RELATED TO THE EINSTEIN WEYL AND ABELIAN HIGGS EQUATIONS

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ABSTRACT. This note describes equations for a pair comprising a Riemannian metric and a Killing field on a surface that contain as special cases the Einstein Weyl equations (in the sense of D. Calderbank) and a real version of a special case of the Abelian Higgs equations, and shows, via the explicit construction of solutions, that the property that a metric solve these equations is preserved by the Ricci flow. Among the metrics arising in this way are the cigar soliton and the sausage metric, and there are other examples of steady gradient Ricci solitons, and eternal, ancient, and immortal Ricci flows, as well as some Ricci flows with conical singularities.

1. INTRODUCTION

Given a surface M , consider, for a pair (h, Y) comprising a Riemannian metric h with scalar curvature \mathcal{R}_h and a vector field Y on M , and a fixed parameter $\varepsilon = \pm 1$, the **real vortex** equations:

$$(1.1) \quad 0 = d(\mathcal{R}_h + 4\varepsilon|Y|_h^2), \quad \mathfrak{L}_Y h = 0.$$

The second condition of (1.1) says simply that Y is a Killing field, while the first equation says that there is a constant τ such that

$$(1.2) \quad \tau = \mathcal{R}_h + 4\varepsilon|Y|_h^2.$$

This parameter τ will be called the **vortex** parameter of the solution (h, Y) . (The factor 4 in (1.1) has no intrinsic significance, as it could be absorbed into Y , and has been chosen for consistency with the conventions of [8]).

A solution (h, Y) of (1.1) will be said to be **trivial** if Y is identically zero. In this case (1.1) forces h to have constant curvature. (The example of a parallel vector field on a flat torus shows that the constancy of the curvature of a solution to (1.1) need not imply the triviality of the solution).

In section 2.1 it is explained how to construct from a solution of (1.1) with $\varepsilon = -1$ a solution of the Einstein Weyl equations as formulated for surfaces by D. Calderbank in [3, 4]. In section 2.2 it is explained that a solution of (1.1) with $\varepsilon = 1$ gives rise to a solution of the usual Abelian Higgs (vortex) equations on the bundle of holomorphic one forms, and can be seen as the real part of such a vortex solution. Because of these observations, a solution (h, Y) to the equations (1.1) will be called **Einstein Weyl** or **vortex like** as $\varepsilon = -1$ or $\varepsilon = 1$. As is also explained in section 2.2 the Einstein Weyl case admits a complex reformulation in which it resembles the Abelian Higgs equations but with a sign change on one term (see (2.3)). These signed Abelian Higgs equations make sense for sections of line bundles other than the tangent bundle, and the corresponding real equations are like (1.1), though with a completely symmetric tensor in place of Y and some appropriate generalization of the Killing condition. While in both the Einstein Weyl and the vortex like case these equations are most interesting when posed for sections of line bundles other than the tangent bundle, the particular case (1.1) considered here is interesting because it can be solved explicitly in quite elementary terms, and the solutions come in one parameter families solving the Ricci flow. A special case of this last statement, explained in section 2.4 and 2.5 (see

Lemma 2.3 in particular) is that in most cases a steady gradient Ricci soliton determines a solution to the Einstein Weyl case of (1.1).

Since the Ricci flow preserves Killing fields, the Killing field Y of a solution (h, Y) to the real vortex equations is preserved by the Ricci flow $h(t)$ beginning with the metric h . While it is not obvious that the real vortex condition (1.2) is also preserved by the Ricci flow, for some τ depending on the flow parameter t , if it is imposed as an auxiliary condition that $(h(t), Y)$ solve (1.1), then the resulting equations can be integrated explicitly. Thus the Ricci flow can somehow be regarded as a flow obtained by varying the vortex parameter, although a precise formulation of this statement and a conceptual explanation for it are not given here. Also, an *a priori* proof that the Ricci flow starting with a solution of (1.1) preserves (1.2) seems to be not so simple, and it is not clear whether such a statement is true in full generality.

Section 3 is devoted to describing in detail the metrics that result. The Ricci flows $h(t)$ arising as solutions to (1.1) include a number of well known examples. In particular they include steady gradient Ricci solitons, such as the cigar soliton; the Fateev-Onofri-Zamolodchikov-King-Rosenau sausage metrics on the sphere; and a pair of solutions to the Ricci flow, one on the sphere and one on the torus, that were found in [8] in connection with Einstein-Weyl structures. Several of the metrics in section 3 have been considered previously by I. Bakas in [1] and can be found by using the ansatz used in [7] to find the sausage metric, but among the metrics constructed there are some apparently new families of gradient Ricci solitons and immortal, ancient, and eternal Ricci flows, and there are many examples having conical singularities. While the examples that were not already well known mostly have some undesirable properties, e.g. bad curvature growth, it is interesting that among the fairly limited class of metrics solving the real vortex equations appear many of the most interesting Ricci flows on surfaces. Also note that the ansatz used in [1] and [7] to find sausage like metrics is here derived, as a consequence of the requirement that the metric be part of a Ricci flow.

The observation made in [8] that Einstein Weyl structures on spheres and tori come in families solving the Ricci flow coupled with the formal similarity between the Einstein Weyl and Abelian vortex equations suggested that there should be a solution of the 2-vortex equations on the sphere expressed in terms of the sausage metric. That this is in fact the case is shown in section 3.15, where all multiplicity free 2-vortices on the sphere are expressed explicitly in terms of the sausage metrics. For the solutions to the Ricci flow arising from Einstein Weyl structures on the sphere and torus, the time parameter can be seen as analogous to the vortex parameter appearing in the Abelian Higgs equations. This suggested on the one hand the convenience of having a common formulation for the Einstein Weyl and Abelian Higgs equations, and on the other hand, that the solutions to these equations should be preserved by Ricci flow.

These observations lend some credence to the idea that the Ricci flow is related to some natural flow obtained from the moduli space of solutions to some vortex-like equations by varying the vortex parameter. In [15], N. Manton has given indications of a possible relation flow obtained by considering the metric on the moduli space of one vortices on a compact Riemann surface as a function of the vortex parameter and the Ricci flow on that surface. Although no direct connection between Manton's ideas and those explained here is yet apparent, they have a similar spirit.

2. REAL VORTEX EQUATIONS

2.1. A pair $(\nabla, [h])$ comprising a torsion-free affine connection ∇ and a conformal structure $[h]$ is a **Weyl structure** if for each $h \in [h]$ there is a one-form γ_i such that $\nabla_i h_{jk} = 2\gamma_i h_{jk}$. Here, and where convenient in all that follows, the abstract index conventions are used; in particular grouping of indices between parentheses (resp. square brackets) indicates complete symmetrization (resp. anti-symmetrization) over the enclosed indices. The one-form γ is the **Faraday primitive** associated to $h \in [h]$, and $F = -d\gamma$ is the **Faraday curvature**. Since when h is rescaled conformally γ

changes by addition of an exact one-form, F depends only on the pair $(\nabla, [h])$ and not on the choice of $h \in [h]$. Note that γ depends only on the homothety class of h and not on h itself. An $h \in [h]$ for which the associated Faraday primitive γ is coclosed will be said to be a **distinguished** representative of $[h]$. From the Hodge decomposition it follows that if the underlying manifold is compact there is a distinguished representative determined uniquely up to homothety.

In dimensions greater than two a Weyl structure is said to be **Einstein** if the trace-free symmetric part of the Ricci tensor of ∇ vanishes. However, since on a surface such a condition is automatic, the situation for surfaces is similar to that for ordinary metrics on surfaces, where the correct analogue of the Einstein condition is constant scalar curvature. In [3] and [4], D. Calderbank defined a Weyl structure $(\nabla, [h])$ on a surface to be **Einstein** if it satisfies the equation

$$(2.1) \quad \begin{aligned} 0 &= \nabla_i(|\det h|^{1/2}(\mathcal{R}_h - 2d_h^*\gamma)) + 2|\det h|^{1/2}h^{pq}\nabla_p F_{iq} \\ &= |\det h|^{1/2}(d_i(\mathcal{R}_h - 2d_h^*\gamma) + 2\gamma_i(\mathcal{R}_h - 2d_h^*\gamma) + 2h^{pq}\nabla_p F_{iq}) \end{aligned}$$

where h is any representative of h , h^{ij} is the symmetric bivector inverse to h_{ij} , and d_h^* is the adjoint of the exterior differential corresponding to h . The equation (2.1) is conformally invariant, for if $\tilde{h} = fh$ then $f(\mathcal{R}_h - 2d_h^*\gamma) = \mathcal{R}_{\tilde{h}} - 2d_{\tilde{h}}^*\tilde{\gamma}$, where $\tilde{\gamma}$ is the Faraday primitive associated to \tilde{h} . The quantity $\mathcal{R}_h - 2d_h^*\gamma$ arises as the h -trace of the Ricci curvature of ∇ . By the following theorem, on a compact surface, Calderbank's Einstein Weyl condition is equivalent to the $\varepsilon = -1$ case of the real vortex equations (1.1). Calderbank's original definition is Definition 3.2 of [3]; see also Corollary 3.4 of that same paper, the conclusion of which was taken as the definition in Definition 6.2 of [8]. For the proof of Theorem 2.1 see Theorem 3.7 of [3] or Theorem 7.1 of [8].

Theorem 2.1 ([3]). *A Weyl structure $(\nabla, [h])$ on a compact surface is Einstein if and only if for any distinguished metric $h \in [h]$ with associated Faraday primitive γ , the vector field $Y^i = h^{ip}\gamma_p$ metrically dual to γ is h -Killing and with h constitutes a solution to the real vortex equations (1.1) in the case $\varepsilon = -1$, that is $d(\mathcal{R}_h - 4|Y|_h^2) = 0$.*

Here it is convenient to take the $\varepsilon = -1$ case of (1.1) as the definition of an Einstein Weyl structure. Precisely, given a pair (h, Y) solving (1.1) with $\varepsilon = -1$, the Weyl connection of the associated Einstein Weyl structure is recovered as $\nabla = D - 2\gamma_{(i}\delta_{j)}^k + h_{ij}Y^k$ where $\gamma_i = Y^ph_{ip}$ and D is the Levi-Civita connection of h .

2.2. The vortex or Abelian Higgs equations on a compact Riemann surface are a modification of the Ginzburg-Landau model for superconductors first studied by M. Noguchi, [16], and S. Bradlow, [2], (see [12] for background and context).

A **Riemann surface** means a one-dimensional complex manifold. It is equivalent to specify a conformal structure $[h]$ and an orientation, in which case the complex structure J is the unique one compatible with $[h]$ and the given orientation. On a Riemann surface a real vector field Y is conformal Killing if and only if its $(1, 0)$ part $Y^{(1,0)}$ is a holomorphic vector field. In particular, the only orientable compact surfaces possibly supporting nontrivial solutions to (1.1) are the sphere and torus. In the Einstein Weyl case, all such structures have been described in various forms in [3], [4], and section 10 of [8].

On a compact Riemann surface with complex structure J let h be a Riemannian metric representing the conformal structure and having Kähler form ω , and let \mathcal{E} be a smooth complex line bundle over M . The Abelian Higgs equations with parameter τ are the following equations for a triple (∇, m, s) , in which m is a Hermitian metric on \mathcal{E} , ∇ is a Hermitian connection on (\mathcal{E}, m) , and s is a smooth section of \mathcal{E} :

$$(2.2) \quad \Omega^{(0,2)} = 0, \quad \bar{\partial}_{\nabla}s = 0, \quad 2i\Lambda(\Omega) + |s|_m^2 = \tau.$$

Here Ω is the curvature of ∇ , viewed as a real-valued two-form on M ; $\bar{\partial}_\nabla$ is the $(1,0)$ part of ∇ ; and Λ is the dual Lefschetz operator on $(1,1)$ forms, normalized so that $\Lambda(\omega) = 1$. The first two equations say that $\bar{\partial}_\nabla$ is a holomorphic structure on \mathcal{E} with respect to which s is a holomorphic section, while the third equation is something like an Einstein equation. A solution of (2.2) is **nontrivial** if s is not identically zero. The trivial solutions correspond to holomorphic structures on \mathcal{E} ; a precise statement is Theorem 4.7 of [2].

The modification of (2.2) to be considered here consists in the equations

$$(2.3) \quad \Omega^{(0,2)} = 0, \quad \bar{\partial}_\nabla s = 0, \quad 2i\Lambda(\Omega) + \varepsilon|s|_m^2 = \tau,$$

in which all the data is as in (2.2), and ε is one of ± 1 . The $\varepsilon = +1$ case simply yields (2.2). The $\varepsilon = -1$ case will be called here the **signed Abelian Higgs equations**.

Solutions to the Abelian Higgs equations are considered equivalent if they are related by the action of the unitary gauge group (S^1 -valued functions) on the space of pairs (∇, s) comprising a Hermitian connection and a smooth section of \mathcal{E} , and the moduli space of solutions means the quotient of the space of pairs solving (2.2) by the action of the unitary gauge group.

The basic theorem about the Abelian Higgs equations on a surface is the following.

Theorem 2.2 ([16], [2], [9]). *Let M be a compact surface equipped with a Kähler metric (h, J) . Let \mathcal{E} be a smooth complex line bundle with a fixed Hermitian metric m . Let D be an effective divisor of degree equal to $\deg(\mathcal{E})$. There exists a nontrivial solution (s, ∇) of the vortex equations (2.2), unique up to unitary gauge equivalence, if and only if $4\pi \deg(\mathcal{E}) < \tau \text{vol}_h(M)$. Moreover the holomorphic line bundle and section canonically associated to D are $(\mathcal{E}, \bar{\partial}_\nabla)$ and s .*

The number $\deg \mathcal{E}$ is called the **vortex number** because the section s has $\deg \mathcal{E}$ zeros (counted with multiplicity), which are regarded as *vortices*. For the same reason, if $\deg \mathcal{E} = N$, a solution (h, s) is referred to as an N -vortex solution, with or without multiplicity as the zeros of s are not or are distinct.

The space of effective divisors on M of a given degree r is the symmetric product $S^r(M)$ of M and Theorem 2.2 shows that $S^{\deg(\mathcal{E})}(M)$ is in bijection with the moduli space of gauge equivalence classes of vortex solutions on \mathcal{E} . It is shown in [9] by symplectic reduction, that this moduli space carries a Kähler structure.

As is explained in [2], a problem equivalent to solving (2.2) consists in finding the Hermitian metric m for which (2.2) hold, given a holomorphic line bundle \mathcal{E} on a Kähler surface and a prescribed section s of \mathcal{E} . From this point of view the complex gauge group of \mathcal{E} (comprising smooth functions $g : M \rightarrow \mathbb{C}^*$) acts by pushforward on holomorphic structures and holomorphic sections, and on Hermitian metrics by multiplication by a factor $|g|^2$. This way of viewing (2.2) is the most relevant for the relation with the real equations (1.1).

On a Riemann surface, a q -differential is a smooth section of the q th power \mathcal{K}^q of the complex cotangent bundle. The real part of a q -differential s is a trace-free symmetric q -tensor X , covariant or contravariant according to whether q is positive or negative, and so s can be written as the $(|q|, 0)$ part of the real tensor X , $s = X^{(|q|, 0)}$. That s be holomorphic is equivalent to X being a Codazzi tensor, for $q > 1$; to X being harmonic, for $q = 1$; and to X being a conformal Killing tensor, for q negative (see Lemma 3.5 of [8]).

Let $\mathcal{E} = \mathcal{K}^q$ for some $q \in \mathbb{Z}$. In the equations (2.3) there is no *a priori* relation between the Kähler metric h on M and the Hermitian metric m on \mathcal{E} . However, since \mathcal{E} is a power of the complex tangent bundle, it makes sense to speak of the Hermitian metric induced on \mathcal{E} by h , and so it makes sense to consider solutions in which m is this induced Hermitian metric. In what follows this will be case of primary interest. In this case a solution s of (2.3) is a holomorphic q -differential and ∇ is the connection induced on \mathcal{E} by the Levi-Civita connection D of h ; the corresponding divisor is canonical, so the solution of (2.3) will be said to be **canonical** as well.

A holomorphic line bundle on $\mathbb{P}^1(\mathbb{C})$ is determined up to isomorphism by its degree. Since \mathcal{K}^q has degree $-2q$, in the case of \mathcal{E} of even degree there is no loss of generality in assuming from the beginning that \mathcal{E} is a power of the complex tangent bundle. In this case the moduli space of q -vortex solutions is the q -fold symmetric product of $\mathbb{P}^1(\mathbb{C})$, which is isomorphic to $\mathbb{P}^q(\mathbb{C})$ via the map associating to a q -tuple of points in $\mathbb{P}^1(\mathbb{C})$ the coefficients of the degree q polynomial vanishing at these points.

A canonical solution to the (signed) vortex equations comprises a Riemann surface (M, h, J) and a holomorphic q -differential s satisfying

$$(2.4) \quad 2i\Lambda(\Omega) + \varepsilon|s|_m^2 = \tau.$$

The Levi-Civita connection D of h induces a Hermitian connection, also denoted by D , on \mathcal{E} . The curvature of D on \mathcal{E} is $\Omega = (iq\mathcal{R}_h/2)\omega_h$, so that $\Lambda(\Omega) = (iq/2)\mathcal{R}_h$. Write $s = 2^{1-q/2}|q|^{1/2}X^{(|q|,0)}$. Then, since $2|X^{(|q|,0)}|_h^2 = |X|_h^2$,

$$(2.5) \quad 2i\Lambda(\Omega) + \varepsilon|s|_k^2 = -q(\mathcal{R}_h - \varepsilon \operatorname{sgn}(q)2^{1-q}|X|_h^2),$$

so that s solves (2.4) if and only if $\mathcal{R}_h - \varepsilon \operatorname{sgn}(q)2^{1-q}|X|_h^2$ is constant. Thus a canonical solution of the (signed) vortex equations is equivalent to a divergence free Codazzi or conformal Killing tensor satisfying

$$(2.6) \quad D_i(\mathcal{R}_h - \varepsilon \operatorname{sgn}(q)2^{1-q}|X|_h^2) = 0.$$

If the Gauss-Bonnet theorem is valid, e.g. if M has finite topological type, finite volume, and integrable curvature, and there is a solution to (2.6), there must hold

$$(2.7) \quad 4\pi\chi(M) = \varepsilon \operatorname{sgn}(q)2^{1-q}\|X\|_h^2 + \tau \operatorname{vol}_h(M),$$

where τ is the constant value of $\mathcal{R}_h - \varepsilon \operatorname{sgn}(q)2^{1-q}|X|_h^2$. This shows that the condition

$$(2.8) \quad \varepsilon \operatorname{sgn}(q) \left(\tau - \frac{4\pi\chi(M)}{\operatorname{vol}_h(M)} \right) \leq 0,$$

on τ is necessary for the existence of solutions.

Note that by themselves the equations (2.6) for (q, ε) and $(-q, -\varepsilon)$ are the same up to a power of 2 that can be absorbed into the section X ; what changes with the change in parameters is the condition (Codazzi or conformal Killing) imposed on the section X . For example, in the $q = \pm 1$ cases, it is different to demand that a vector field be the real part of a holomorphic vector field and that its dual one-form be the real part of a holomorphic differential; the former imposes that the vector field be conformal Killing while the latter imposes that the dual one-form be harmonic.

The case most important here is $q = -1$, in which case s is a holomorphic vector field and so X is a conformal Killing field. In this case solutions of the real vortex equations (1.2) give rise to solutions of the signed Abelian Higgs equations. However, not all solutions of the signed Abelian Higgs equations arise in this way because it is not the case that every holomorphic vector field is the $(1, 0)$ part of a real Killing field. The real version of (2.6) is then (1.2), but with Y only required to be conformal Killing.

In general, if a symmetric trace-free $|q|$ -tensor X is given such that $X^{(|q|,0)}$ is holomorphic, then the equation (2.6) can be solved as follows. Let \tilde{h} be the unique metric conformal to h with scalar curvature contained in $\{0, 2, -2\}$ and write $h = e^u \tilde{h}$. The equation (2.6) becomes

$$(2.9) \quad \Delta_{\tilde{h}} u - \mathcal{R}_{\tilde{h}} + \tau e^u + \varepsilon \operatorname{sgn}(q)2^{1-q}e^{(1-q)u}|X|_{\tilde{h}}^2 = 0.$$

The solvability or no of (2.9) on a compact surface of genus at least two is usually ascertainable on general grounds, while on spheres, tori, and some noncompact surfaces it is less straightforward. Particularly for the sphere and torus it can be more convenient to use as the background metric the singular flat metric ${}^*h = |X|^{2/q}h$ instead of \tilde{h} . For example, on a surface of genus at least two, the

case $\varepsilon = -1$, $q \geq 1$ of (2.9) can always be solved (see Corollary 9.1 of [8]) provided τ is negative. For another example, in the $\varepsilon = -1$ and $q = -1$ case X is a Killing field, and using this fact the analogue of (2.9) with *h in place of \tilde{h} reduces to an ordinary differential equation which can be solved straightforwardly. Cast in different terms, this is essentially the approach used in section 3 below.

The equation (2.9) is slightly more complicated than the similar equation arising for the vortex equations, e.g. equation 4.1 of [2]. The difference is the requirement that the metric on \mathcal{E} be that induced by the metric on the underlying surface. This condition introduces an extra term in (2.9), with the consequence that solvability of (2.9) does not reduce directly to the well known results of Kazdan-Warner, [13]. Other cases of (2.9) have been studied previously. For example, the case with $\varepsilon = 1$, $q = 2$, and $\tau < 0$ arises as the Gauss equation for a minimal surface in a hyperbolic three manifold; see Theorem 4.2 of [19], in which this equation plays an important role.

2.3. Clearly two solutions (h, Y) and (\bar{h}, \bar{Y}) of the real vortex equations (1.1) related by a diffeomorphism of the underlying manifold should be considered isomorphic (geometrically equivalent). In solving (1.1) on an orientable manifold it is convenient to fix the underlying complex structure, and to fix the vector field Y , and to look for a metric h representing the conformal structure determined by the complex structure and such that (h, Y) solves (1.1). This corresponds to the equivalent formulation of the problem of solving the Abelian Higgs equations, in which the holomorphic structure and the section s are fixed, described in the preceding section. Fixing the complex structure reduces the possible isomorphisms of a solution (h, Y) to those induced by biholomorphisms of the underlying complex manifold. From the point of view usually used in regards to the Abelian Higgs equations, solutions (h, Y) related by a biholomorphisms would *not* be identified, because the zeros of s are regarded as vortices, and their absolute position is considered meaningful. This should be kept in mind particularly when M has a large biholomorphism group, e.g. when M is the Riemann sphere. On the other hand, it means that to solve (1.1) on the Riemann sphere, it suffices to consider a single Y representing a given orbit in the space of Killing fields under the action of the biholomorphism group, for the solutions corresponding to other Killing fields in the orbit of Y can then be obtained by pullback.

On the other hand, the notion of equivalence relevant for the Abelian Higgs equations is gauge equivalence. How this translates for pairs (h, Y) is described now. Since the underlying complex structure has been fixed, the gauge transformations fixing the induced holomorphic structure on a power of the canonical bundle are just multiplications by a given $z = e^c e^{i\theta} \in \mathbb{C}^*$. The corresponding action on pairs (h, Y) is $z \cdot (h, Y) = (|z|^2 h, |z|^{-2} \text{Re}(e^{-i\theta} Y^{(1,0)})) = (e^{2c} h, e^{-2c} Y^\theta)$, where $Y^\theta = \cos \theta Y + \sin \theta JY$ is the real part of $e^{-i\theta} Y^{(1,0)}$. In general this action does not preserve the property that Y be Killing. A precise statement is the following.

Lemma 2.1. *Let M be a surface with a Kähler structure (h, J) and let $Y \in \Gamma(TM)$ be a Killing field for h . If for some $\theta \in (0, 2\pi)$ the vector field $Y^\theta = \cos \theta Y + \sin \theta JY$ is Killing for h then either Y is parallel or $\theta = \pi$ and $Y^\theta = -Y$.*

Proof. Suppose Y and Y^θ are Killing for some $\theta \in (0, 2\pi)$. Let γ be the one-form dual to Y ; then $\star \gamma$ is the one-form dual to JY . Since Y is Killing, γ is coclosed. Since Y and Y^θ are Killing, so is $\sin \theta JY = Y^\theta - \cos \theta Y$. If $\sin \theta \neq 0$ this means JY is Killing, and so $2D \star \gamma = d \star \gamma = 0$, the last equality because γ is coclosed. In this case JY is parallel, and so Y is parallel. Hence if both Y and Y^θ are Killing for some $\theta \in (0, 2\pi)$ then either Y is parallel, or $\theta = \pi$ and $Y^\theta = -Y$. \square

It follows that if (h, Y) solves (1.1) then so too does $\pm e^c \cdot (h, Y) = (e^{2c} h, \pm e^{-2c} Y)$ for all $c \in \mathbb{R}$. Since in general homothetic metrics need not be diffeomorphic, it is at first not clear in what sense the solutions $e^c \cdot (h, Y)$ and (h, Y) are equivalent. The Levi-Civita connection of $e^{2c} h$ does not depend on c . Since also the one-form γ dual to $e^{-2c} Y$ via $e^{2c} h$ does not depend on c , the resulting

Weyl connection is independent of c . Hence, in the case $\varepsilon = -1$, corresponding to the Einstein Weyl equations, the solutions $e^c \cdot (h, Y)$ and (h, Y) are equivalent in the sense that they determine the same Einstein Weyl structure. This justifies regarding (h, Y) and $e^c \cdot (h, Y)$ as equivalent even though they are not related by a diffeomorphism, and it is natural to extend this notion of equivalence to the $\varepsilon = 1$ case as well. Solutions equivalent in this sense will be said to be **gauge equivalent**, for, as was explained in the preceding paragraphs, this equivalence is a vestigial manifestation of the gauge equivalence of the associated solutions of the signed Abelian Higgs equations.

It is also the case that if (h, Y) solves (1.1) then so too does $(h, -Y)$, and these solutions could also be considered gauge equivalent. Note that it can happen that (h, Y) and $(h, -Y)$ be equivalent modulo a diffeomorphism if there is an isometry of h sending Y to $-Y$, but in general there need not be such an isometry. On the other hand, in the $\varepsilon = -1$ case, (h, Y) and $(h, -Y)$ generally induce nonisomorphic Weyl structures.

2.4. On a surface a **Ricci soliton** is a Riemannian metric h for which there are a vector field X^i and a constant $c \in \mathbb{R}$ such that $\frac{1}{2}\mathcal{R}_h h_{ij} + \frac{1}{2}(\mathfrak{L}_X h)_{ij} = ch_{ij}$. It is a **gradient Ricci soliton** if X^i is the h -gradient of a smooth function f . In this case $\mathcal{R}_h + \Delta_h f = 2c$. Differentiating this and using the Ricci identity shows that $d\mathcal{R}_h = \mathcal{R}_h df$, from which it follows that $\mathcal{R}_h + |df|_h^2 - 2cf$ is constant. In particular, in the case the gradient Ricci soliton is steady, meaning $c = 0$, the pair $(h, \frac{1}{2}h^{ip}df_p)$ satisfies the first equation of (1.1).

The gradient $X^i = h^{ip}df_p$ of the potential of a gradient Ricci soliton is conformal Killing, for by definition $\mathfrak{L}_X h = (2c - \mathcal{R}_h)h$. If h is a steady gradient Ricci soliton, it follows that the pair $(h, \frac{1}{2}X)$ solves the variant of (1.1) in which the vector field must be conformal Killing. If, moreover, there is θ such that $X^\theta = \cos \theta X + \sin \theta JX$ is Killing, where J is the complex structure determined by h and a given orientation of the surface, then $(h, \frac{1}{2}X^\theta)$ is a solution of the vortex like equations (1.1).

By Theorem 10.1 of [10], any Ricci soliton on a compact surface has constant curvature, so no interesting examples of solutions to (1.1) arise in this way on compact surfaces. On the other hand, there are explicit examples of steady Ricci solitons on noncompact surfaces, and these yields nontrivial explicit solutions of (1.1), as will be detailed in section 3. For example, by Theorem 26.3 of [11], a complete Ricci soliton on a surface having bounded curvature assuming somewhere its maximum is diffeomorphic to the cigar soliton (see section 3.8 below for details). Hamilton's way of constructing the cigar soliton in [10] also yields examples on orbifolds, as is elaborated upon by L.-F. Wu in [20]. Note also that Hamilton's characterization of the cigar soliton has been improved by P. Daskalopoulos and N. Sesum who in [6] proved that a complete ancient Ricci flow on a surface must be a cigar soliton if it has bounded positive curvature and bounded width (in a sense defined in [6]).

2.5. In this section there are proved some general facts about solutions to (1.1) that will be used in section 3 to find explicit solutions. In the particular case of Einstein Weyl structures, most of discussion appears in some equivalent form in sections 5, 6 of [8]; see in particular Lemma 6.4 of [8].

On an oriented surface M consider a pair (h, Y) . Let J be the complex structure determined by h and the given orientation and let ω_h be the Kähler form of h . Define a one-form γ by $\gamma = \iota(Y)h$. The Hodge star on one-forms is given by $\star\alpha = -\alpha \circ J$. Write $F = -d\gamma$ and define a function \mathcal{F}_h by $2F = \mathcal{F}_h \omega_h$ (equivalently, $2\star F = \mathcal{F}_h$).

If Y is a Killing field, then γ is coclosed, so $d\star\gamma = 0$. Let \tilde{M} be the universal cover of M . The pullbacks to \tilde{M} of objects defined on M will be written with the same notation. On \tilde{M} there is a globally defined function μ such that $d\mu = -\star\gamma$. By definition $|\gamma|_h^2 \omega_h = \gamma \wedge \star\gamma = d\mu \wedge \gamma$. Interior multiplying with Y shows that $|\gamma|_h^2 (\iota(Y)\omega_h + d\mu) = 0$. Since Y is the real part of a holomorphic vector field, its zeros are isolated. Hence the zeros of the pullback to \tilde{M} of γ are isolated as well,

and the preceding identity implies $\iota(Y)\omega_h + d\mu = 0$ on \tilde{M} . This means that μ is a moment map for the action generated by the vector field on \tilde{M} dual to the pullback of γ .

Now suppose (h, Y) solves the real vortex equations (1.1). Since Y is Killing the dual one-form $\gamma_i = Y^p h_{ip}$ satisfies $4D\gamma = 2d\gamma = -\mathcal{F}_h \omega_h$. Observe that $Y^p \omega_{ip} = \gamma_p J_i{}^p = -(\star\gamma)_i$. There follows $D_i|Y|_h^2 = D_i|\gamma|_h^2 = 2Y^p D_i \gamma_p = \frac{1}{2}\mathcal{F}_h(\star\gamma)_i$. Hence the first equation of (1.2) can be rewritten as

$$(2.10) \quad d\mathcal{R}_h = -4\varepsilon d|\gamma|_h^2 = -2\varepsilon \mathcal{F}_h \star \gamma.$$

The full curvature of D is $R_{ijkl} = R_{ijk}{}^p h_{pl} = \mathcal{R}_h h_{l[i} h_{j]k}$ and so $\omega^{ij} R_{ijkl} = -\mathcal{R}_h \omega_{kl}$. Applying this and the Ricci identity yields

$$(2.11) \quad \begin{aligned} D_i \mathcal{F}_h &= D_i(\omega^{pq} F_{pq}) = -\omega^{pq} D_i d\gamma_{pq} = -2\omega^{pq} D_i D_p \gamma_q = -2\omega^{pq} D_p D_i \gamma_q + 2\omega^{pq} R_{i[pq]}{}^a \gamma_q \\ &= 2\omega^{pq} D_{[p} D_{q]} \gamma_i - \omega^{pq} R_{pqi}{}^a = -2\omega^{pq} R_{pqi}{}^a = 2\mathcal{R}_h J_i{}^p \gamma_p = -2\mathcal{R}_h(\star\gamma)_i, \end{aligned}$$

which proves

$$(2.12) \quad d\mathcal{F}_h = -2\mathcal{R}_h \star \gamma.$$

Consequently,

$$(2.13) \quad d(\mathcal{R}_h^2 - \varepsilon \mathcal{F}_h^2) = 2\mathcal{R}_h d\mathcal{R}_h - 2\varepsilon \mathcal{F}_h d\mathcal{F}_h = -4\varepsilon \mathcal{R}_h \mathcal{F}_h \star \gamma + 4\varepsilon \mathcal{F}_h \mathcal{R}_h \star \gamma = 0.$$

Hence $\mathcal{R}_h^2 - \varepsilon \mathcal{F}_h^2$ is constant on M . It will be convenient to call this constant σ :

$$(2.14) \quad \sigma = \mathcal{R}_h^2 - \varepsilon \mathcal{F}_h^2.$$

Actually, something stronger can be said. Let λ be 1 or i as ε is 1 or -1 , so that $\lambda^2 = \varepsilon$. Then, by (2.10) and (2.12),

$$(2.15) \quad \begin{aligned} d(e^{-2\lambda\mu}(\mathcal{R}_h + \lambda\mathcal{F}_h)) &= e^{-2\lambda\mu} (d\mathcal{R}_h + \lambda d\mathcal{F}_h + 2\lambda(\mathcal{R}_h + \lambda\mathcal{F}_h) \star \gamma) \\ &= e^{-2\lambda\mu} (d\mathcal{R}_h + 2\varepsilon \mathcal{F}_h \star \gamma + \lambda(d\mathcal{F}_h - 2\mathcal{R}_h \star \gamma)) = 0, \end{aligned}$$

showing that $e^{-2\lambda\mu}(\mathcal{R}_h + \lambda\mathcal{F}_h)$ is constant on \tilde{M} . Since, in the case that $\lambda = 1$, $\mathcal{R}_h + \lambda\mathcal{F}_h$ is defined on M , and μ is by definition real-valued, it follows that μ descends to a function defined on M , which is a moment map for the group action generated by Y . On the other hand, in the case that $\lambda = i$, it in principle could be that μ is multi-valued when viewed as a function on M .

The preceding is summarized in the following lemma

Lemma 2.2. *Let M be a surface equipped with a Riemannian metric h and an h -Killing field Y such that $\mathcal{R}_h + 4\varepsilon|Y|_h^2$ is constant on M , where $\varepsilon = \pm 1$. Then $\mathcal{R}_h^2 - \varepsilon \mathcal{F}_h^2$ is constant on M and the function $e^{-2\lambda\mu}(\mathcal{R}_h + \lambda\mathcal{F}_h)$ is constant on the universal cover \tilde{M} , where μ is a moment map for the group action on \tilde{M} induced by Y , and λ is 1 or i as ε is 1 or -1 .*

In the $\varepsilon = 1$ case the conclusion of Lemma 2.2 implies that $e^{-2\mu}(\mathcal{R}_h + \mathcal{F}_h)$ is constant, from which it follows that $\mathcal{R}_h + \mathcal{F}_h$ has a definite sign if it is not identically zero. This also means that μ is well defined on M since on \tilde{M} it has the expression $\mu = \frac{1}{2} \log(\frac{\mathcal{R}_h + \mathcal{F}_h}{c})$ for some constant c . In this case Lemma 2.2 also applies to the pair $(\bar{h}, \bar{Y}) = (h, -Y)$ for which $\mathcal{F}_{\bar{h}} = \mathcal{F}_h$ and $\bar{\mu} = -\mu$, so that $\mathcal{R}_h - \mathcal{F}_h$ has also a definite sign if it is not identically zero.

In the case $\varepsilon = -1$ it follows from Lemma 2.2 that the functions $\cos(2\mu)\mathcal{R}_h + \sin(2\mu)\mathcal{F}_h$ and $\cos(2\mu)\mathcal{F}_h - \sin(2\mu)\mathcal{R}_h$ are constant on \tilde{M} .

Some further conclusions can be deduced from (2.10) and (2.12). Namely, a critical point of \mathcal{R}_h (resp. \mathcal{F}_h) is either a zero of \mathcal{F}_h (resp. \mathcal{R}_h) or a zero of Y . In the latter case it is also a critical point of \mathcal{F}_h (resp. \mathcal{R}_h).

Suppose that (h, Y) solves (1.1) and let D be the Levi-Civita connection D of h . Using $4D\gamma = 2d\gamma = -\frac{1}{4}\omega$ it can be checked that

$$(2.16) \quad -Dd\mu = D \star \gamma = \frac{1}{4}\mathcal{F}_h h,$$

in which the first equality makes sense when μ is well-defined. Using (2.16) it is straightforward to show that $4D_Y Y = \mathcal{F}_h JY$ from which it follows that D is given by

$$(2.17) \quad D_Y Y = -\frac{1}{4}\mathcal{F}_h JY, \quad D_{JY} Y = D_Y JY = JD_Y Y = \frac{1}{4}\mathcal{F}_h Y, \quad D_{JY} JY = JD_{JY} Y = \frac{1}{4}\mathcal{F}_h JY.$$

It follows from (2.10) and (2.17) that the unit norm vector field $U = |Y|_h^{-1} JY$, defined on the open dense complement M^* of the zero locus of Y , satisfies $D_U U = 0$, so that its nontrivial integral curves are h -geodesics.

Rewriting (2.16) yields

$$(2.18) \quad \pm 2Dd\mu + \frac{1}{2}\mathcal{R}_h h = \frac{1}{2}(\mathcal{R}_h \pm \mathcal{F}_h)h,$$

so that h is a gradient Ricci soliton with potential $\pm 2\mu$ if $\mathcal{R}_h \pm \mathcal{F}_h$ is constant. In particular, if $\sigma = 0$ then h is a steady gradient Ricci soliton with potential $\pm 2\mu$. Precisely:

Lemma 2.3. *For a solution (h, Y) of the real vortex equations (1.1), the following are equivalent:*

- (1) $\sigma = 0$.
- (2) *One of $\mathcal{R}_h \pm \mathcal{F}_h$ is constant.*
- (3) *One of $\mathcal{R}_h \pm \mathcal{F}_h$ vanishes identically.*

In this case, either $\varepsilon = -1$ and h is a flat metric and Y is a parallel vector field or $\varepsilon = 1$ and h is a steady gradient Ricci soliton with potential $\pm 2\mu$.

Proof. Obviously (3) implies (2). Suppose there holds (2). Then, by (2.10) and (2.12),

$$(2.19) \quad 0 = d(\mathcal{R}_h \pm \mathcal{F}_h) = \mp 2(\mathcal{R}_h \pm \varepsilon \mathcal{F}_h) \star \gamma.$$

Pairing (2.19) with $\star \gamma$ shows that $(\mathcal{R}_h \pm \varepsilon \mathcal{F}_h)|\gamma|_h^2 = 0$, and since γ has isolated zeros this forces $\mathcal{R}_h \pm \varepsilon \mathcal{F}_h = 0$. If $\varepsilon = 1$ this gives $\mathcal{R}_h = \mp \mathcal{F}_h$, and so $\sigma = \mathcal{R}_h^2 - \mathcal{F}_h^2 = 0$. If $\varepsilon = -1$ it gives $\mathcal{R}_h = \pm \mathcal{F}_h$, which means that $2\mathcal{R}_h = \mathcal{R}_h \pm \mathcal{F}_h$ is a constant. By the assumption that (h, Y) solve (1.2), this means that $|\gamma|_h^2$ is constant. If γ has a zero, then it must be identically 0, in which case $\mathcal{F}_h = 0$, and so also $\mathcal{R}_h = 0$ and $\sigma = 0$. If γ has no zero, since Y is Killing, there holds $\mathcal{R}_h = -\Delta_h \log |Y|_h^2$ and this vanishes since $|Y|_h^2$ is constant, so h is flat and $\pm \mathcal{F}_h = \mathcal{R}_h = 0$, and hence also $\sigma = 0$. This shows that (2) implies (1). Suppose $\sigma = 0$. If $\varepsilon = -1$ then, by definition, $0 = \sigma = \mathcal{R}_h^2 + \mathcal{F}_h^2$, so $\mathcal{R}_h = 0 = \mathcal{F}_h$ vanish identically. Hence h is flat and γ , and therefore also Y , is parallel. If $\varepsilon = 1$, then $\mathcal{R}_h^2 = \mathcal{F}_h^2$. In particular one of $\mathcal{R}_h \pm \mathcal{F}_h$ vanishes. This shows that (1) implies (3). Finally, as $\mathcal{R}_h = \mp \mathcal{F}_h$, by (2.18), h is a steady gradient Ricci soliton with potential $\pm 2\mu$. \square

Since, by Theorem 10.1 of [10], a gradient Ricci soliton on a compact surface has constant curvature, on a compact surface, for a solution (h, Y) of the real vortex equations (1.2) having $\sigma = 0$, the metric h must have constant curvature. By (1.2) this implies that Y has constant norm, so either Y is identically zero, or Y has no zeros. In the latter case the argument in the proof of Lemma 2.3 shows that h must be flat and Y must be parallel. However, later it will be shown that in the noncompact case gradient Ricci solitons give rise to nontrivial solutions of the equations (1.1).

Lemma 2.3 could be taken as indicating that solutions to (1.1) are a sort of generalization of Ricci solitons. This would mean that they could be viewed as the fixed points of some natural flow on the moduli space of pairs (h, Y) . It would be interesting to give some substance to such a speculation.

3. RICCI FLOWS SOLVING THE REAL VORTEX EQUATIONS

The result of Lemma 2.3, showing that certain solutions of the real vortex equations are steady gradient Ricci solitons, suggests some relation between the real vortex equations and the Ricci flow. In this section the real vortex equations are solved and it is shown, by explicit construction, that the Ricci flow preserves solutions. In any context in which there is known uniqueness of its solutions, e.g. on compact manifolds, the Ricci flow preserves isometries in the sense that any isometry of the initial metric is an isometry of metrics later in the flow. In particular, a Killing field for the initial metric will be a Killing field all along the flow. What is not self evident *a priori* is that the Ricci flow preserves also the compatibility condition between the metric and the Killing field imposed by (1.1).

3.1. The following conventions are to be understood whenever reference is made to the two sphere \mathbb{S}^2 . Let x and y be coordinates on \mathbb{R}^2 in which the standard metric of scalar curvature 2 and volume 4π on \mathbb{S}^2 has the expression

$$(3.1) \quad h_0 = \frac{4(dx^2 + dy^2)}{(1 + \rho^2)^2} = \frac{2(dr^2 + ds^2)}{1 + \cosh 2s} = \frac{dr^2 + ds^2}{\cosh^2 s},$$

where $\rho^2 = x^2 + y^2$, and the coordinates $s \in \mathbb{R}$ and $r \in \mathbb{R}/2\pi\mathbb{Z}$ on $\mathbb{R}^2 \setminus \{0\}$ are defined by $x = e^s \cos r$ and $y = e^s \sin r$. Here $z = x + iy = e^s e^{ir}$ is a standard coordinate on the Riemann sphere, and the metric h_0 has in the complementary chart with coordinates $\tilde{x} + i\tilde{y} = -1/z = -e^{-s} e^{ir}$ the same expression, with x and y replaced by \tilde{x} and \tilde{y} . The coordinates (r, s) will be referred to as **cylindrical**, as the metric $dr^2 + ds^2$ in these coordinates is the metric on the standard flat cylinder. These coordinates can also be considered as coordinates on the plane, viewed as the universal cover of the cylinder, and will be used in this form in what follows.

For $\beta > -1$, a metric h on a surface M is said to have a **conical singularity** of angle $2\pi(\beta + 1)$ at a point $p \in M$ if there is an open neighborhood $U \subset M$ of p and a diffeomorphism mapping U into \mathbb{C} such that p is mapped to 0 and there is a smooth function f such that the pullback to U of the metric $e^f |z|^{2\beta} |dz|^2 = e^{2(\beta+1)s} (dr^2 + ds^2)$ is equal to h on $U \setminus \{p\}$. If the same condition is satisfied but with $\beta = -1$, then h is said to have a logarithmic singularity or **cusp** at p .

3.2. That a metric $h = u(r, s)(dr^2 + ds^2)$ be rotationally symmetric is equivalent to the requirement that u not depend on r . More generally, if h admits a Killing field then locally it can be put in this form with the Killing field ∂_r . To be precise, suppose M is an orientable surface equipped with a Riemannian metric h and a compatible complex structure J , and let Y be a nontrivial Killing field. Since Y is the real part of a holomorphic vector field, its zeros are isolated. Let M^* be the open dense subset of M on which Y is nonvanishing. Since $\mathcal{L}_Y J = 0$, there holds $[Y, JY] = 0$. If r and s are parameters for the flows generated by Y and JY , then on the domain of their common definition h can be written in the form $h = u(s)(dr^2 + ds^2)$, where that u does not depend on r follows from the fact that $Y = \partial_r$ is Killing. If Y and JY are complete then their flows exist for all time, and r and s can be viewed as global coordinates on the universal cover of M^* . By construction $J\partial_r = -\partial_s$, so that with respect to the coordinate $z = x + iy = e^s e^{ir}$, J is the standard complex structure on \mathbb{C} , and $\partial_r = x\partial_y - y\partial_x$ and $\partial_s = x\partial_x + y\partial_y$. It follows that the Kähler form of the metric $h = u(dr^2 + ds^2)$ consistent with this complex structure is $\omega_h = u ds \wedge dr$.

For a metric g of the form $g = u(r, s)(dr^2 + ds^2)$, the scalar curvature of g is

$$(3.2) \quad \mathcal{R}_g = -u^{-1} \Delta_{\text{euc}} \log u,$$

where Δ_{euc} means the Laplacian of the Euclidean metric $dr^2 + ds^2$. In particular, in the present setting $\mathcal{R}_h = -u^{-1}(\log u)_{ss}$. The one-form γ dual to Y has in these coordinates the form $\gamma = u dr$, and its Hodge dual the form $\star\gamma = -u ds$. In these coordinates, $F = -(\log u)_s \omega_h$, so $\mathcal{F}_h = -2(\log u)_s$. In (2.17) this gives an explicit expression for the Christoffel symbols of D with respect to the frame

$\{\partial_r, \partial_s\}$. Since $d\mu(Y) = \gamma(JY) = 0$, in local coordinates the moment map μ depends only on s and not on r . Precisely, $\mu_s = u$.

Now suppose (h, Y) solves the real vortex equations (1.1). Let r and s be local coordinates on an open set contained in the complement M^* of the zero set of Y such that $Y = \partial_r$ and $JY = \partial_s$. Write \mathcal{R} and \mathcal{F} in lieu of \mathcal{R}_h and \mathcal{F}_h . In local coordinates (2.10) becomes $\mathcal{R}_s = 2\varepsilon u \mathcal{F}$. On the other hand, $2u\mathcal{R} = -2(\log u)_{ss} = \mathcal{F}_s$. The corresponding intrinsic statement is (2.12). The geodesic vector field U defined just after (2.17) is expressible in local coordinates as $U = u^{-1/2}\partial_s$.

3.3. Let (h, Y) solve (1.2). Subtracting the square of $\mathcal{R}_h = \tau - 4\varepsilon|Y|_h^2$ from $\sigma = \mathcal{R}_h^2 - \varepsilon\mathcal{F}_h^2$ yields

$$(3.3) \quad \sigma - \tau^2 = -\varepsilon\mathcal{F}_h^2 - 8\varepsilon\tau|Y|_h^2 + 16|Y|_h^4.$$

In local coordinates in which the pair (h, Y) has the form $(u(s)(dr^2 + ds^2), \partial_r)$, (3.3) becomes

$$(3.4) \quad \sigma - \tau^2 = -4\varepsilon(\log u)_s^2 - 8\varepsilon\tau u + 16u^2.$$

Since $u > 0$ there can be defined $w = u^{-1}$ and in terms of w , (3.4) becomes

$$(3.5) \quad w_s^2 - 4\varepsilon + 2\tau w - \rho w^2 = 0,$$

where the constant ρ is defined by $\rho = \varepsilon(\tau^2 - \sigma)/4$. Differentiating (3.5) shows that, where $w_s \neq 0$, there holds

$$(3.6) \quad w_{ss} = \rho w - \tau,$$

the general solution of which is of course quite simple. From (3.6) it is to be expected that the geometric properties of (h, Y) will depend in part on whether ρ vanishes, is positive, or is negative. The general solution of (3.6) has two free parameters, but (3.5) imposes on them a further relation, leaving a single degree of freedom.

Solutions (h, Y) to (1.2) can be constructed by reversing the preceeding. Given σ and τ , one solves (3.5) for w , defines $u = w^{-1}$, and defines $h = u(dr^2 + ds^2)$ and $Y = \partial_r$. Whether the resulting solution extends when $s \rightarrow \pm\infty$ has to be analyzed on a case by case basis.

3.4. Some subtleties arise when rescalings are considered. Replacing (h, Y) by $(\bar{h}, \bar{Y}) = (e^c h, e^{-c} Y)$ replaces τ and σ by $e^{-c}\tau$ and $e^{-2c}\sigma$, so replaces ρ by $e^{-2c}\rho$. In the preceeding the parameters r and s were determined by the flows of JY and Y , and so upon rescaling are replaced by the parameters $\bar{r} = e^c r$ and $\bar{s} = e^c s$ corresponding to $e^{-c}JY$ and $e^c Y$. It is straightforward to check that if $\bar{w}(\bar{s})$ is the function obtained from (\bar{h}, \bar{Y}) as was w obtained from (h, Y) , then $\bar{w}(s) = e^c w(e^{-c}s)$, so that \bar{w} solves (3.5) with $\bar{\tau}$ and $\bar{\sigma}$ in place of τ and σ if and only if w solves (3.5). In this sense, by rescaling (h, Y) the parameter ρ can be normalized to take a given value, e.g. 0, 4, or -4 . Precisely, if (h, Y) is given, it determines a ρ , and there is a gauge equivalent pair (\bar{h}, \bar{Y}) determining $e^c \rho$, and so ρ can be rescaled as one likes. However, such a rescaling presupposes that the gauge equivalence class of (h, Y) is already known. If instead (3.5) is to be solved in order to construct (h, Y) by inverting the procedure used to derive (3.5), then the particular value of ρ may matter because the parameter s has implicitly been fixed. For instance, if the metric resulting from the solution of (3.5) is to be extended to some manifold containing the cylinder as a subset there has to be analyzed whether a rescaling of ρ can be achieved via a geometric or gauge equivalence of the resulting structure on this larger manifold.

3.5. In what follows it will be shown that when (h, Y) solves the vortex like equations (1.2) then $(h(t), Y)$ solves these equations where $h(t)$ is the Ricci flow through h . Since $e^{-c}h(e^c t)$ solves the Ricci flow if $h(t)$ does, it follows that also $(e^{-c}h(e^c t), e^c Y)$ solves (1.2) for any $c \in \mathbb{R}$.

That a one-parameter family of metrics $h(t) = u(s, t)(dr^2 + ds^2)$ moreover evolve by the Ricci flow $\frac{d}{dt}h = -\mathcal{R}_h h$ is equivalent to u solving the **logarithmic diffusion equation**

$$(3.7) \quad u_t = (\log u)_{ss}.$$

In terms of the function $w = u^{-1}$, equation (3.7) becomes

$$(3.8) \quad w_t = ww_{ss} - w_s^2.$$

Suppose that $(h(t), Y(t))$ is a one-parameter family of solutions to the real vortex equations. Combining (3.5), (3.6), and (3.8) shows that for $h(t)$ also to be a solution to the Ricci flow necessitates

$$(3.9) \quad w_t = w(\rho w - \tau) - 4\varepsilon + 2\tau w - \rho w^2 = \tau w - 4\varepsilon,$$

in which τ is a function of t . It will be seen that in all cases the equation (3.9) can be satisfied.

In what follows there are analyzed the possible solutions of (3.6) and the resulting Ricci flows. Though the geometrically meaningful parameters are ε , τ , and σ , it is often more convenient to write explicit expressions in terms of the derived parameter $\rho = \varepsilon(\tau^2 - \sigma)/4$. It will also be convenient to let λ be 1 or i as ε is 1 or -1 , so that $\varepsilon = \lambda^2$. Also, not all possible combinations of (λ, τ, σ) can be realized.

3.6. The soliton case: $\sigma = 0$ and $\varepsilon = -1$. The first case considered is $\sigma = 0$. By Lemma 2.3, the solutions with $\varepsilon = -1$ have h flat and Y parallel. The more interesting case with $\varepsilon = 1$ is referred to as the *soliton case* because, by Lemma 2.3, the solutions with $\varepsilon = 1$ are steady gradient Ricci solitons. In this case, since $\tau^2 = 4\varepsilon\rho = 4\rho$, either $\tau = 0 = \rho$, or $\rho > 0$ and $\tau = \pm 2\sqrt{\rho}$.

Consider the case $\tau = 0 = \rho$. By (3.5), $w_s^2 = 4$, and so there is a constant b such that $w = \pm 2s + b$. The resulting solution (h, Y) will be defined on the half space in the (r, s) -plane where w is positive and, as will be apparent from the special case (3.10) below, will not be smoothly extendible to any larger domain. Replacing s by its translate $s - q$ or its reflection $-s$ yields an isomorphic solution on a different half-space, so there is no loss of generality in supposing that $a = 2$ and $b = 0$, in which case the metric

$$(3.10) \quad h = \frac{1}{2s}(dr^2 + ds^2) = \frac{|dz|^2}{2|z|^2 \log|z|},$$

is defined on the half cylinder where $s > 0$, or, equivalently, the complement $\{z \in \mathbb{C} : |z| > 1\}$ of the unit disk in the plane. The metric (3.10) blows up as $z \rightarrow \infty$, so extends smoothly to no larger domain, as asserted above. The curvature $\mathcal{R}_h = -\mathcal{F}_h = -2s^{-1}$ is negative on the entire domain of h , but is not bounded from below. By Lemma 2.3, the metric (3.10) is a steady gradient Ricci soliton with potential equal to twice the moment map $\mu = \frac{1}{2} \log s = \frac{1}{2} \log |z|$.

There remains the case $\sigma = 0$ and $\rho = \tau^2/4 > 0$. Equation (3.5) reduces to $w_s^2 = \frac{\tau^2}{4}(w - \frac{\tau}{4})^2$ which has the general solution $w(s) = 4/\tau + ae^{\pm \tau s/2}$ for some $a \in \mathbb{R}$. The case $a = 0$ determines a flat metric h with a parallel Y , so it can be assumed $a \neq 0$.

Since an equivalent geometric structure is obtained upon replacing s by $-s$, in this case it can be supposed that w has the form $w = 4/\tau + ae^{-\tau s/2}$ (choosing the negative sign turns out to be convenient). Similarly, by a translation in s , the value of a can be normalized to take any given nonzero value having the same sign as the original a . However, with such a normalization of a there is not obtained a Ricci flow through solutions of (1.2) because setting a to a nonzero constant is not compatible with (3.21). Regarding τ and a as functions of t and substituting $w = 4/\tau + ae^{-\tau s/2}$ into (3.9) yields the equations $\tau_t = 0$ and $a_t = \tau a$, so that τ is constant and $a(t)$ can be taken to have the form $a(t) = \epsilon \tau^{-1} e^{\tau(t-t_0)}$ for $\epsilon = \pm 1$. In this case a pair $(h(t), Y)$ solving (1.2) and for which the metrics $h(t)$ constitute a Ricci flow is given by

$$(3.11) \quad h(t) = u(dr^2 + ds^2) = \frac{\tau(dr^2 + ds^2)}{4(1 + \epsilon e^{\tau(t-t_0)} e^{-\tau s/2})} = \frac{\tau|z|^{\tau/2-2}|dz|^2}{4(|z|^{\tau/2} + \epsilon e^{\tau(t-t_0)})},$$

where the last expression is valid when it makes sense. It remains to determine the dependence on ϵ and τ , and the largest domains of definition of the resulting metric. If $\epsilon = 1$ then it must be that $\tau > 0$, in which case u is defined at least in the entire (r, s) plane. If $\epsilon = -1$ then u is positive on the half cylinder $\text{sgn}(\tau)(s - 2(t - t_0)) > 0$.

In the cases with $\tau > 0$ it is relevant to consider what happens to $h(t)$ when $s \rightarrow \infty$. To this end, take $\tilde{z} = -z^{-1}$; in this coordinate the metric $h(t)$ has the form

$$(3.12) \quad \frac{\tau |d\tilde{z}|^2}{4|\tilde{z}|^2(1 + \epsilon e^{\tau(t-t_0)}|\tilde{z}|^{\tau/2})},$$

and so has a cusp when $\tilde{z} = 0$, that is when $s \rightarrow \infty$.

Suppose now $\epsilon = 1$. When $s \rightarrow -\infty$ ($z = 0$), it follows from the last expression of (3.11) that $h(t)$ has at $z = 0$ a conical singularity with angle $\pi\tau/2$. In particular, in the case $\tau = 4$ the metric $h(t)$ extends smoothly across $z = 0$ (when $s \rightarrow -\infty$); the resulting metric is the cigar soliton.

Regard the plane with coordinates (r, s) as the universal cover of the cylinder (punctured plane). Consider the metric $h(t)$ defined as in (3.11) on the plane with coordinates (r, s) and consider the corresponding metric $\tilde{h}(\tilde{t})$, defined in the same fashion on the plane with coordinates (\tilde{r}, \tilde{s}) in place of (r, s) , with \tilde{r} in place of r , and with time parameters \tilde{t} and \tilde{t}_0 . Let $Y = \partial_s$ and $\tilde{Y} = \partial_{\tilde{s}}$. Suppose $\tilde{r} = 1$ and define a diffeomorphism by $(\tilde{r}, \tilde{s}) = \phi(r, s) = (\tau r, \tau s)$. Set $\tilde{t} = \tau t$ and $\tilde{t}_0 = \tau t_0$. It is easily checked that $\phi^*(h(t)) = \tau^{-1}\tilde{h}(\tilde{t})$ and $\phi^*(Y) = \tau\tilde{Y}$. Hence the pulled back pair $\phi^*(h(t), Y) = (\phi^*h(t), \phi^*Y) = (\tau^{-1}\tilde{h}(\tilde{t}), \tau\tilde{Y})$ is gauge equivalent to the pair $(\tilde{h}(\tilde{t}), \tilde{Y})$. This shows that the solutions on the plane given by (3.11) are equivalent for different values of τ . On the other hand, if these solutions are regarded as metrics with conical singularities on the cylinder, then they are not equivalent, because when viewed as a map on the punctured plane ϕ is not a diffeomorphism and does not in general extend smoothly through the puncture.

In the case $\epsilon = -1$ and $\tau > 0$, the metric (3.11) is defined on the half cylinder $s > 2(t - t_0)$. Equivalently, it is defined on $|z| > e^{2\tau(t-t_0)}$. As explained before, $h(t)$ has a cusp when $s \rightarrow \infty$. In the case $\epsilon = -1$ and $\tau < 0$, the metric (3.11) is defined on the half cylinder $s < 2(t - t_0)$. Equivalently, it is defined on the punctured disk $0 < |z| < e^{2\tau(t-t_0)}$. To analyze the behavior as $s \rightarrow -\infty$ it is convenient to rewrite the last expression of (3.11) as

$$(3.13) \quad h(t) = \frac{-\tau |dz|^2}{4|z|^2(e^{\tau(t-t_0)}|z|^{-\tau/2} - 1)},$$

from which it is apparent that $h(t)$ has a cusp at the origin.

In all cases there hold

$$(3.14) \quad \mathcal{R}_{h(t)} = -\mathcal{F}_{h(t)} = \frac{\tau \epsilon e^{\tau(t-t_0)}}{e^{\tau s/2} + \epsilon e^{\tau(t-t_0)}} = \frac{\tau}{1 + \epsilon e^{-\tau(t-t_0)} e^{\tau s/2}} = \frac{\tau \epsilon e^{\tau(t-t_0)}}{|z|^{\tau/2} + \epsilon e^{\tau(t-t_0)}}.$$

If $\epsilon = 1$ then $\tau > \mathcal{R}_h > 0$ with the limiting values τ and 0 approached as $s \rightarrow -\infty$ and $s \rightarrow \infty$, respectively. If $\epsilon = -1$ then \mathcal{R}_h is negative, approaching 0 as $\text{sgn}(\tau)s \rightarrow \infty$ and approaching $-\infty$ as $s \rightarrow 2(t - t_0)$.

3.7. Cases with $\sigma \neq 0$ and $\rho = 0$. When $\sigma \neq 0$ it is convenient to separate the cases $\rho = 0$ and $\rho \neq 0$. In this section, suppose $\sigma \neq 0$ and $\rho = 0$, so that $\tau^2 = \sigma \neq 0$.

By (3.6), $w = -\frac{\tau}{2}s^2 + as + b$ for some real constants a and b . From (3.5) it follows that $4\epsilon = a^2 + 2\tau b$. Replacing s by a translate $s - q$, it may be supposed that $a = 0$, so that $b = 2\epsilon\tau^{-1}$ and $w = -\frac{\tau}{2}(s^2 - \frac{4\epsilon}{\tau^2})$. In this case the equation (3.9) becomes $\tau_t = \tau^2$.

It follows from (3.3) that if $\epsilon = -1$ and $\tau > 0$ then Y is identically 0, so this case can be excluded. Hence if $\epsilon = -1$ it can be supposed $\tau < 0$. In this case w has no real roots, so is positive for all $s \in \mathbb{R}$. The equation (3.9) becomes $\tau_t = \tau^2$, which has the solutions $\tau(t) = (t_0 - t)^{-1}$ that are

negative for $t > t_0$. In particular, taking $t_0 = 0$ there results the one-parameter family of metrics,

$$(3.15) \quad h(t) = \frac{2t(dr^2 + ds^2)}{s^2 + 4t^2} = \frac{2t|dz|^2}{|z|^2((\log|z|)^2 + 4t^2)},$$

defined for all $s \in \mathbb{R}$ and all $t > 0$. From the expression (3.15) it is apparent that $h(t)$ blows up as $s \rightarrow \pm\infty$, so that $h(t)$ can be viewed as a metric on the infinite cylinder or the punctured plane. The metrics (3.15) constitute an immortal solution to the Ricci flow such that for each $t > 0$ the pair $(h(t), Y = \partial_r)$ solves the Einstein Weyl equations (1.2). Since $\tau = -t^{-1}$, the curvature of $h(t)$ is

$$(3.16) \quad \mathcal{R}_{h(t)} = \tau - 4\varepsilon|Y|_{h(t)}^2 = -t^{-1} + 4u = \frac{4t^2 - s^2}{t(s^2 + 4t^2)}.$$

Since $u \leq 1/(2t)$, it follows that $-t^{-1} < \mathcal{R}_{h(t)} \leq t^{-1}$, with equality on the righthand side when $s = 0$. The curvature is positive when $|s| < 2t$ and negative when $|s| > 2t$.

Consider the case $\varepsilon = 1$. In this case w has the real roots $\pm 2\tau^{-1}$. First suppose $\tau > 0$. Then w is positive for $-2\tau^{-1} < s < 2\tau^{-1}$. The equation $\tau_t = \tau^2$ has the solutions $\tau(t) = -(t - t_0)^{-1}$ that are positive for $t < t_0$. In particular, if $t_0 = 0$ then $\tau = -t^{-1}$ and there results the family of metrics,

$$(3.17) \quad h(t) = \frac{-2t(dr^2 + ds^2)}{4t^2 - s^2} = \frac{-2t|dz|^2}{|z|^2(4t^2 - (\log|z|)^2)},$$

defined for $t < 0$ and $2t < s < -2t$. The strip $|s| < -2t$ is a bounded open cylinder or an annulus in the plane. This is an ancient solution to the Ricci flow such that for each $t < 0$ the pair $(h(t), Y = \partial_r)$ solves the real vortex equations (1.2). The curvature of $h(t)$ is

$$(3.18) \quad \mathcal{R}_{h(t)} = \tau - 4\varepsilon|Y|_{h(t)}^2 = -t^{-1} - 4u = \frac{4t^2 + s^2}{t(4t^2 - s^2)}.$$

Since $u \geq -1/(2t)$, with equality when $s = 0$, it follows that $\mathcal{R}_{h(t)} \leq t^{-1}$, with equality when $s = 0$, and that $\mathcal{R}_h \rightarrow -\infty$ when $s \rightarrow \pm 2t$, so that \mathcal{R}_h is strictly negative and unbounded from below.

Finally, suppose $\varepsilon = 1$ and $\tau < 0$. Then w is positive for $|s| > -2\tau^{-1}$. The equation $\tau_t = \tau^2$ has the solutions $\tau(t) = -(t - t_0)^{-1}$ that are negative for $t > t_0$. In particular, if $t_0 = 0$ then $\tau = -t^{-1}$ and there results a family of metrics having the form (3.17), though defined for $t > 0$ and $|s| > 2t$. From (3.17) it is apparent that the metric $h(t)$ blows up when $s \rightarrow \pm\infty$, so that the maximal domain of definition of $h(t)$ is the disconnected region $|s| > 2t$ which is the disjoint union of two half-infinite cylinders, or the disjoint union of the punctured disk and the complement of a disk in the plane. This is an immortal solution to the Ricci flow. Its curvature is as in (3.18).

3.8. Generalities related to the cases where $\sigma \neq 0$ and $\rho \neq 0$. When $\sigma \neq 0$ and $\rho \neq 0$ the most general real solution to (3.6) can be written in the form

$$(3.19) \quad w = \tau/\rho + a \cosh \sqrt{\rho}s + b\sqrt{\operatorname{sgn}(\rho)} \sinh \sqrt{\rho}s,$$

where a and b are real constants, $\sqrt{\operatorname{sgn}(\rho)}$ means 1 or i as ρ is positive or negative, $\sqrt{\rho}$ means $\sqrt{\operatorname{sgn}(\rho)}\sqrt{|\rho|}$, and a and b satisfy the conditions imposed by the positivity of u . Substituting (3.19) into (3.5) yields

$$(3.20) \quad \sigma = \rho^2(a^2 - \operatorname{sgn}(\rho)b^2).$$

In particular, the assumption $\sigma \neq 0$ precludes the case $a = 0 = b$ (which yields the flat metric on the plane), so it is the case that not both a and b vanish.

3.8.1. Suppose w is as in (3.19) and that τ , σ , and ρ are functions of time, while ϵ is constant. Requiring that w satisfy (3.9) yields a family of metrics $h(t)$ evolving by Ricci flow. While *a priori* ρ depends on t , if it is supposed that ρ is constant in t , then (3.9) reduces to the easily solved equations

$$(3.21) \quad \tau_t = \tau^2 - 4\epsilon\rho = \sigma, \quad a_t = \tau a, \quad b_t = \tau b.$$

Using (3.20), the second and third equations of (3.21) can be combined to yield $\sigma_t = 2\tau\sigma$. Let λ be 1 or i as ϵ is 1 or -1 , so that $\lambda^2 = \epsilon$. Solving the first equation of (3.21) shows that τ and σ may be taken to have the forms

$$(3.22) \quad \tau(t) = 2\lambda\sqrt{\rho}\coth(2\lambda\sqrt{\rho}(t_0 - t)), \quad \sigma(t) = 4\epsilon\rho\operatorname{csch}^2(2\lambda\sqrt{\rho}(t_0 - t)), \quad \text{or}$$

$$(3.23) \quad \tau(t) = 2\sqrt{\rho}\tanh(2\sqrt{\rho}(t_0 - t)), \quad \sigma(t) = -4\rho\operatorname{sech}^2(2\sqrt{\rho}(t_0 - t)).$$

In (3.23) the parameters ϵ and λ are omitted because when $\sigma < 0$ they must take the values $\epsilon = 1 = \lambda$.

Solving for a and b yields metrics $h(t)$ that evolve by the Ricci flow. The explicit expressions for the resulting metrics are not written yet, as it will be convenient to make some parameter dependent normalizations beforehand. If $\tau(t)$ as in (3.22) or (3.23) and the corresponding a and b are substituted in (3.19) the resulting pair $(h(t), Y)$ solves (1.2) while the metrics $h(t)$ constitute a Ricci flow. The qualitative behavior of the resulting solutions depends on the values and signs of σ , ρ , λ , and ϵ . In particular, it is convenient to consider the cases $\sigma > 0$ and $\sigma < 0$ separately.

3.8.2. The ansatz

$$(3.24) \quad u = 2\lambda^2(a(t) + b(t)\cosh 2\lambda s)^{-1},$$

for solutions of (3.7), in which λ is either 1 or i and $a(t)$ and $b(t)$ are real functions defined on some connected open subset of \mathbb{R} , was used by Fateev-Onofri-Zamolodchikov in [7] to find solutions to the Ricci flow (regarded in [7] as the one-loop approximation to the renormalization group flow). In [7] the extra generality of the parameter λ was not needed because solutions of this sort could be excluded by physical considerations. Note here that the form (3.24), or, rather, the slightly more general equivalent form (3.19) for $w = u^{-1}$, is derived rather than postulated, and what is of interest is that solutions to (1.1) necessarily come in families parameterized by Ricci flows.

3.8.3. That ρ is constant along the solutions $(h(t), Y)$ that will be constructed means that the right-hand side of (3.3) is constant in t , something which is not apparent *a priori*, and appears to be complicated to prove directly.

3.8.4. Note that the metrics (3.11) can be obtained as the $a = \pm b$ case of (3.19). Excluding the case $\sigma = 0 = \tau$, if $\sigma = 0$ then $\epsilon\rho = \tau^2 > 0$, in which case solving the first equation of (3.21) yields

$$(3.25) \quad \tau(t) = \pm 2\lambda\sqrt{\rho}, \quad \sigma(t) = 0.$$

The cases (3.25) arise as the limits of (3.22) when $t_0 \rightarrow \pm\infty$.

3.9. **Cases with $\sigma > 0$.** If $\sigma > 0$ then, by (3.20), $a^2 - \operatorname{sgn}(\rho)b^2$ is positive, and there is a real number q such that

$$(3.26) \quad \begin{aligned} w &= \tau/\rho + a\cosh\sqrt{\rho}s + b\sqrt{\operatorname{sgn}(\rho)}\sinh\sqrt{\rho}s \\ &= \tau/\rho + (a^2 - \operatorname{sgn}(\rho)b^2)^{1/2}\cosh(\sqrt{\rho}s + \sqrt{\operatorname{sgn}(\rho)}q) \\ &= \rho^{-1}\left(\tau + \epsilon\sqrt{\sigma}\cosh(\sqrt{\rho}s + \sqrt{\operatorname{sgn}(\rho)}q)\right). \end{aligned}$$

The factor $\epsilon = \pm 1$ comes from the choice of a square root of ρ^2 . If $\rho > 0$, then q is unique, and if $\rho < 0$, then q is unique if it is chosen from $[0, 2\pi)$. The expression (3.26) handles both the $\rho > 0$ and $\rho < 0$ cases simultaneously.

Since an equivalent geometric structure is obtained upon translation of s , it can be supposed that $q = 0$, or, equivalently, that $b = 0$, so that in this case w has the form

$$(3.27) \quad w = \rho^{-1}(\tau + \epsilon\sqrt{\sigma} \cosh \sqrt{\rho}s).$$

It then follows that if $\tau(t)$ is as in (3.22) then the resulting pair $(h(t), Y)$ solves (1.2) while the metrics $h(t)$ constitute a Ricci flow. Explicitly

$$(3.28) \quad \begin{aligned} h(t) &= \frac{\sqrt{\rho} \sinh(2\lambda\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{2\lambda (\cosh(2\lambda\sqrt{\rho}(t_0 - t)) + \epsilon \cosh \sqrt{\rho}s)} \\ &= \frac{\epsilon\sqrt{\rho} \sinh(2\lambda\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}-2}|dz|^2}{\lambda (|z|^{2\sqrt{\rho}} + 2\epsilon \cosh(2\lambda\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1)} \end{aligned}$$

The expression (3.28) encodes in principle various qualitatively different metrics, depending on the values of the various parameters. In particular it is convenient to separate the cases $\rho > 0$ and $\rho < 0$. These cases are detailed separately in sections 3.10 and 3.11.

Note that, in the cases where it makes sense, when the coordinate z is replaced by $\tilde{z} = -z^{-1}$ the form of the last expression in (3.28) is unchanged. From this expression it follows that, when it makes sense, the metric $h(t)$ has at the origin $z = 0$ or at the point at infinity a conical singularity with angle $\pi\sqrt{\rho}$.

3.9.1. The scalar curvature of the metric $h(t)$ of (3.28) is

$$(3.29) \quad \begin{aligned} \mathcal{R}_{h(t)} &= \frac{\sigma + \epsilon\tau\sqrt{\sigma} \cosh \sqrt{\rho}s}{\tau + \epsilon\sqrt{\sigma} \cosh \sqrt{\rho}s} = \frac{\epsilon\sqrt{\sigma}((\tau/\sqrt{\sigma}) \cosh \sqrt{\rho}s + \epsilon)}{\tau/\sqrt{\sigma} + \epsilon \cosh \sqrt{\rho}s} \\ &= \frac{2\epsilon\lambda\sqrt{\rho}}{\sinh(2\lambda\sqrt{\rho}(t_0 - t))} \frac{\cosh(2\lambda\sqrt{\rho}(t_0 - t)) \cosh \sqrt{\rho}s + \epsilon}{\cosh(2\lambda\sqrt{\rho}(t_0 - t)) + \epsilon \cosh \sqrt{\rho}s}. \end{aligned}$$

Similarly,

$$(3.30) \quad \mathcal{F}_{h(t)} = \frac{2\epsilon\sqrt{\sigma}\sqrt{\rho} \sinh \sqrt{\rho}s}{\tau + \epsilon\sqrt{\sigma} \cosh \sqrt{\rho}s} = \frac{2\epsilon\sqrt{\rho} \sinh \sqrt{\rho}s}{\cosh 2\lambda\sqrt{\rho}(t_0 - t) + \epsilon \cosh \sqrt{\rho}s}.$$

3.9.2. From (2.17) it follows that $D_{JY}JY \wedge JY = 0$, so that the integral curves of ∂_s , that is the radial curves with r constant, are projective geodesics, meaning their images coincide with the images of $h(t)$ -geodesics. Similarly, from (2.17) it follows that $D_Y Y$ vanishes along the zero locus of $\mathcal{F}_{h(t)}$. The zero locus of \mathcal{F}_h is $s = 0$ when $\rho > 0$, and integer multiples of $\pi/\sqrt{|\rho|}$ when $\rho < 0$. Hence when these curves are contained in the domain of $h(t)$ they are $h(t)$ -geodesics.

3.10. Cases with $\sigma > 0$ and $\rho > 0$.

3.10.1. In the cases in which $\lambda = 1$ it follows from $\sigma = \mathcal{R}_{h(t)}^2 - \mathcal{F}_{h(t)}^2$ that $\mathcal{R}_{h(t)}^2$ has a maximum wherever $\mathcal{F}_{h(t)}$ vanishes. From (3.28) it is apparent that, even when $h(t)$ extends as $s \rightarrow \pm\infty$, the zero locus of $\mathcal{F}_{h(t)}$ is $s = 0$, provided $s = 0$ is in the domain of $h(t)$. Hence in this case $|\mathcal{R}_{h(t)}| \leq \sqrt{\sigma}$, and equality is obtained exactly when $s = 0$.

3.10.2. *Case $\rho > 0$, $\lambda = 1$, $\epsilon = 1$.* The metrics

$$(3.31) \quad h(t) = \frac{\sqrt{\rho} \sinh(2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{2(\cosh(2\sqrt{\rho}(t_0 - t)) + \cosh \sqrt{\rho}s)} = \frac{\sqrt{\rho} \sinh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}-2}|dz|^2}{|z|^{2\sqrt{\rho}} + 2 \cosh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1}$$

constitute an ancient Ricci flow, being defined for $t \in (-\infty, t_0)$. They extend to all of \mathbb{S}^2 with conical singularities of angle $\pi\sqrt{\rho}$ at the origin and the point at infinity (equivalently as $s \rightarrow \pm\infty$), which are the zeros of Y . In the particular case $\rho = 4$, the metrics $h_{sau}(t) = h(t)$ extend smoothly to the entire two sphere. These metrics $h_{sau}(t)$ are often called the *King-Rosenau* metrics because the corresponding solutions of the logarithmic diffusion equation were found by P. Rosenau in [17] and J. R. King in [14]. These metrics were found independently by V. Fateev, E. Onofri, and A. B. Zamolodchikov in [7], and the more descriptive appellation *sausage metric* given by the authors of [7] to these metrics is used here.

According to the main theorem of [5], an ancient solution Ricci flow on a compact surface which is not a shrinking soliton is equivalent via a diffeomorphism to the sausage metrics $h_{sau}(t)$.

By (3.28), the curvature of (3.31) is

$$(3.32) \quad \mathcal{R}_{h(t)} = \frac{2\sqrt{\rho}}{\sinh(2\sqrt{\rho}(t_0 - t))} \frac{\cosh(2\sqrt{\rho}(t_0 - t)) \cosh \sqrt{\rho}s + 1}{\cosh 2\sqrt{\rho}(t_0 - t) + \cosh \sqrt{\rho}s},$$

which is strictly positive and extends smoothly to all of \mathbb{S}^2 . It is claimed that

$$(3.33) \quad 0 < 2\sqrt{\rho} \operatorname{csch}(2\sqrt{\rho}(t_0 - t)) = \sqrt{\sigma(t)} \leq \mathcal{R}_{h(t)} \leq \tau(t) = 2\sqrt{\rho} \coth(2\sqrt{\rho}(t_0 - t)).$$

Since $\mathcal{R}_{h(t)}^2 = \sigma + \mathcal{F}_{h(t)}^2 \geq \sigma$, there holds $\mathcal{R}_{h(t)} \geq \sqrt{\sigma}$, with equality exactly when $\mathcal{F}_{h(t)} = 0$, which occurs on the equatorial geodesic $s = 0$. Since $\mathcal{R}_h \leq \mathcal{R}_h + 4|Y|_h^2 = \tau$, there holds $\mathcal{R}_h \leq \tau$, with equality exactly where Y vanishes, which occurs at the cone points.

By (3.33) the family of homothetic metrics

$$(3.34) \quad k(t) = \tau(t)h(t) = \frac{\rho \cosh(2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{\cosh(2\sqrt{\rho}(t_0 - t)) + \cosh \sqrt{\rho}s} = \frac{2\rho \cosh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}-2}|dz|^2}{|z|^{2\sqrt{\rho}} + 2 \cosh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1}$$

has curvature satisfying $0 < \operatorname{sech} 2\sqrt{\rho}(t_0 - t) \leq \mathcal{R}_{k(t)} \leq 2$. As $t \rightarrow t_0$ the metrics $k(t)$ converge pointwise to the constant curvature ρ metric

$$(3.35) \quad \frac{\rho(dr^2 + ds^2)}{2 \cosh^2(\sqrt{\rho}s/2)} = \frac{\rho(dr^2 + ds^2)}{1 + \cosh \sqrt{\rho}s} = \frac{2\rho|z|^{\sqrt{\rho}-2}|dz|^2}{(1 + |z|^{\sqrt{\rho}})^2},$$

defined on \mathbb{S}^2 with two cone points of angle $\pi\sqrt{\rho}$. As $t \rightarrow -\infty$, the metrics $k(t)$ converge pointwise to the flat metric $\rho(dr^2 + ds^2)$ on the punctured plane.

3.10.3. *Case $\rho > 0$, $\lambda = i$, $\epsilon = \pm 1$.* It will be shown now that in the cases in which $\sigma > 0$, $\rho > 0$, and $\lambda = i$, since $\cos(\pi - x) = -\cos(x)$ and $\sin(\pi - x) = \sin(x)$, the two cases obtained by taking $\epsilon = 1$ or $\epsilon = -1$ are equivalent up to an orientation-reversing unimodular transformation of t , and so it suffices to consider the case $\epsilon = 1$. In the case $\epsilon = 1$, the metrics

$$(3.36) \quad h(t) = \frac{\sqrt{\rho} \sin(2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{2(\cos(2\sqrt{\rho}(t_0 - t)) + \cosh \sqrt{\rho}s)} = \frac{\sqrt{\rho} \sin(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}-2}|dz|^2}{|z|^{2\sqrt{\rho}} + 2 \cos(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1}$$

are defined for $t \in (t_0 - \pi/(2\sqrt{\rho}), t_0)$. They extend to all of \mathbb{S}^2 with conical singularities of angle $\pi\sqrt{\rho}$ at the origin and the point at infinity (equivalently as $s \rightarrow \pm\infty$), which are the zeros of Y . In the particular case $\rho = 4$, the metrics $h_{ew}(t) = h(t)$ extend smoothly to the entire two sphere.

Temporarily write $h^-(t)$ for the metric obtained from (3.28) with $\sigma > 0$, $\rho > 0$, $\lambda = i$, parameter t_0 , and $\epsilon = -1$. The correspondence between $h(t)^-$ and $h(t)$ is then given by

$$(3.37) \quad \begin{aligned} h^-(t) &= \frac{-\sqrt{\rho} \sin(2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{2(\cos(2\sqrt{\rho}(t_0 - t)) - \cosh \sqrt{\rho}s)} = \frac{\sqrt{\rho} \sin(\pi - 2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{2(\cos(\pi - 2\sqrt{\rho}(t_0 - t)) + \cosh \sqrt{\rho}s)} \\ &= \frac{\sqrt{\rho} \sin(2\sqrt{\rho}(t - t_0 + \pi/(2\sqrt{\rho}))(dr^2 + ds^2)}{2(\cos(2\sqrt{\rho}(t - t_0 + \pi/(2\sqrt{\rho})) + \cosh \sqrt{\rho}s)} = h(-t + 2t_0 - \pi/(2\sqrt{\rho})). \end{aligned}$$

Notice that as a consequence there must hold $\mathcal{R}_{h^-(t)} = -\mathcal{R}_{h(-t+2t_0-\pi/(2\sqrt{\rho}))}$, which can also be verified directly using (3.29). Although henceforth there is considered only the metric $h(t)$ of the case $\epsilon = 1$, it is a special property of the Ricci flow $h(t)$ that it remains a Ricci flow under time reversal.

By (3.29), the curvature of (3.36) is

$$(3.38) \quad \mathcal{R}_{h(t)} = \frac{2\sqrt{\rho}}{\sin(2\sqrt{\rho}(t_0 - t))} \frac{\cos(2\sqrt{\rho}(t_0 - t)) \cosh \sqrt{\rho}s + 1}{\cos(2\sqrt{\rho}(t_0 - t)) + \cosh \sqrt{\rho}s}.$$

The curvature of $h(t)$ satisfies

$$(3.39) \quad 2\sqrt{\rho} \cot(2\sqrt{\rho}(t_0 - t)) = \tau(t) \leq \mathcal{R}_{h(t)} \leq \sqrt{\sigma(t)} = 2\sqrt{\rho} \csc(2\sqrt{\rho}(t_0 - t)).$$

Although $h(t)$ has strictly positive curvature if $t \in (t_0 - \pi/(4\sqrt{\rho}), t_0)$, it has some negative curvature for $t \in (t_0 - \pi/(2\sqrt{\rho}), t_0 - \pi/(4\sqrt{\rho}))$. The curvature attains its maximum along the equatorial geodesic $s = 0$ where $\mathcal{F}_{h(t)}$ vanishes, and tends to its minimum at the cone points, when $s \rightarrow \pm\infty$.

Precisely, for $t \in (t_0 - \pi/(2\sqrt{\rho}), t_0 - \pi/(4\sqrt{\rho}))$ the curvature is positive exactly on the equatorial band

$$(3.40) \quad |s| < \frac{1}{\sqrt{\rho}} \log \tan(\sqrt{\rho}(t_0 - t) - \pi/4).$$

To check this, observe that from (3.38) it is apparent that \mathcal{R}_h is positive exactly where $\cosh \sqrt{\rho}s < -\sec(2\sqrt{\rho}(t_0 - t))$. Solving this inequality for $e^{\sqrt{\rho}s}$ yields the equivalent set of inequalities

$$(3.41) \quad -\sec(2\sqrt{\rho}(t_0 - t)) - \tan(2\sqrt{\rho}(t_0 - t)) < e^{\sqrt{\rho}s} < -\sec(2\sqrt{\rho}(t_0 - t)) + \tan(2\sqrt{\rho}(t_0 - t)).$$

Simplifying (3.41) using the trigonometric identity $\tan((a+b)/2) = (\sin a + \sin b)/(\cos a + \cos b)$ with $b = \pi/2$ and $a = 2\sqrt{\rho}(t_0 - t)$ yields (3.40).

The family of homothetic metrics

$$(3.42) \quad k(t) = \sqrt{\sigma(t)}h(t) = \frac{\rho(dr^2 + ds^2)}{\cos(2\sqrt{\rho}(t_0 - t)) + \cosh \sqrt{\rho}s} = \frac{2\rho|z|^{\sqrt{\rho}-2}|dz|^2}{|z|^{2\sqrt{\rho}} + 2\cos(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1}$$

has curvature satisfying $-2 \leq \cos(2\sqrt{\rho}(t - t_0)) \leq \mathcal{R}_{k(t)} \leq 2$. As $t \rightarrow t_0$ the metrics $k(t)$ tend to a constant curvature ρ metric (3.35) on \mathbb{S}^2 with two cone points of angle $\pi\sqrt{\rho}$, while when $t \rightarrow t_0 - \pi/(2\sqrt{\rho})$ the metrics $k(t)$ converge pointwise to the constant curvature $-\rho$ metric

$$(3.43) \quad \frac{\rho(dr^2 + ds^2)}{2\sinh^2(\sqrt{\rho}s/2)} = \frac{\rho(dr^2 + ds^2)}{\cosh \sqrt{\rho}s - 1} = \frac{2\rho|z|^{\sqrt{\rho}-2}|dz|^2}{(1 - |z|^{\sqrt{\rho}})^2},$$

which is defined on the disks complementary to the equator $s = 0$ and has cone points of angle $\pi\sqrt{\rho}$ at the centers of these disks. In the particular case $\rho = 4$ the metrics $k(t)$ interpolate between the spherical metric and the hyperbolic metric.

3.10.4. In the preceeding two cases, where $\sigma > 0$, $\rho > 0$, and $\epsilon = 1$, and in which $h(t)$ is defined on \mathbb{S}^2 , the geodesic vector field $U = |Y|_{h(t)}^{-1} JY$ defined in section 2.5 is defined on the punctured place $\mathbb{C} \setminus \{0\}$. The integral curves of U are $h(t)$ -geodesics. Their images are contained in the usual great circles (lines of longitude) passing through the poles (the cone points) on \mathbb{S}^2 , although they are not parameterized as geodesics of the round metric. Likewise, the equatorial circle $s = 0$, where $\mathcal{F}_{h(t)}$ vanishes, is an $h(t)$ -geodesic.

3.10.5. *Case $\rho > 0$, $\lambda = 1$, $\epsilon = -1$.* For the metrics

$$(3.44) \quad h(t) = \frac{\sqrt{\rho} \sinh(2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{2(\cosh(2\sqrt{\rho}(t_0 - t)) - \cosh \sqrt{\rho}s)} = \frac{-\sqrt{\rho} \sinh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}-2}|dz|^2}{|z|^{2\sqrt{\rho}} - 2 \cosh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1}$$

there are two cases, distinguished by the sign of $\tau(t) = 2\sqrt{\rho} \coth(2\sqrt{\rho}(t_0 - t))$. When $t < t_0$, $\tau(t)$ is positive, and $h(t)$ is defined on the bounded open cylinder $2(t_0 - t) > |s|$. When $t > t_0$, $\tau(t)$ is negative and $h(t)$ is defined on $2(t_0 - t) < |s|$, and has cone points with angle $\pi\sqrt{\rho}$ when $s \rightarrow \pm\infty$, so is the disjoint union of two infinite pointed cigarlike manifolds (they are smooth when $\rho = 4$). In the $\tau < 0$ case the map sending s to $-s$ (equivalently z to $-z^{-1}$) extends to an involution interchanging the two connected components.

In both cases the curvature is

$$(3.45) \quad \mathcal{R}_{h(t)} = \frac{2\sqrt{\rho}}{\sinh(2\sqrt{\rho}(t_0 - t))} \frac{1 - \cosh(2\sqrt{\rho}(t_0 - t)) \cosh \sqrt{\rho}s}{\cosh(2\sqrt{\rho}(t_0 - t)) - \cosh \sqrt{\rho}s},$$

which is strictly negative on the domain of definition of $h(t)$. The curvature is not bounded from below for it tends to $-\infty$ as $|s| \rightarrow 2|t_0 - t|$. Since $0 < \sigma = \mathcal{R}_{h(t)}^2 - \mathcal{F}_{h(t)}^2 \leq \mathcal{R}_{h(t)}^2$ and $\mathcal{R}_{h(t)}$ is negative, $\mathcal{R}_{h(t)}$ is bounded from above by $-\sqrt{\sigma}$, and assumes its maximum at any point at which $\mathcal{F}_{h(t)}$ vanishes, at which point its value is $-\sqrt{\sigma}$. The latter occurs if $s = 0$. Hence, in the $\tau > 0$ case, the curvature of $h(t)$ satisfies

$$(3.46) \quad \mathcal{R}_h \leq -\sqrt{\sigma} = -2\sqrt{\rho} \operatorname{csch}(2\sqrt{\rho}(t_0 - t)),$$

with equality along the equatorial geodesic $s = 0$. Since $\sigma = \tau^2 + 4\rho \geq \tau^2$, when $\tau < 0$ there holds $\tau < -\sqrt{\sigma}$. In this case, since $\tau = \mathcal{R}_{h(t)} + 4|Y|_{h(t)}^2 \geq \mathcal{R}_{h(t)}$, there holds

$$(3.47) \quad \mathcal{R}_h \leq \tau = 2\sqrt{\rho} \coth(2\sqrt{\rho}(t_0 - t)),$$

with equality when $s \rightarrow \pm\infty$, that is, at the cone points.

When $\tau(t) < 0$ the family of homothetic metrics

$$(3.48) \quad k(t) = \sqrt{\sigma}h(t) = \frac{\rho \operatorname{sgn}(t_0 - t)(dr^2 + ds^2)}{\cosh(2\sqrt{\rho}(t_0 - t)) - \cosh \sqrt{\rho}s} = \frac{-2\rho \operatorname{sgn}(t - t_0)|z|^{\sqrt{\rho}-2}|dz|^2}{|z|^{2\sqrt{\rho}} - 2 \cosh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1}$$

converges pointwise, as t tends to t_0 from above, to the constant curvature $-\rho$ singular metric (3.43) on the disjoint union of two disks. On the other hand, when t tends to t_0 from below, the metrics $k(t)$ collapse to a point. When $\tau(t) > 0$ the family of homothetic metrics

$$(3.49) \quad \tilde{k}(t) = \tau(t)h(t) = \frac{\rho \cosh(2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{\cosh(2\sqrt{\rho}(t_0 - t)) - \cosh \sqrt{\rho}s} = \frac{-2\rho \cosh(2\sqrt{\rho}(t - t_0))|z|^{\sqrt{\rho}-2}|dz|^2}{|z|^{2\sqrt{\rho}} - 2 \cosh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} + 1}$$

converges pointwise, as t tends to t_0 from above, to the flat metric $\rho(dr^2 + ds^2)$ on the infinite cylinder.

3.11. Cases with $\sigma > 0$ and $\rho < 0$. In considering the cases in which $\sigma > 0$ and $\rho < 0$ it is convenient to note that, since $\cosh \sqrt{\rho}s = \cos \sqrt{|\rho|}s = -\cos(\sqrt{|\rho|}s + \pi)$, the subcases $\epsilon = 1$ and $\epsilon = -1$ are equivalent modulo a translation in s , and so there is no loss of generality in normalizing ϵ to be one of ± 1 , as is convenient. Here it is chosen to normalize $\epsilon = -1$ in these cases, which are the next two considered.

3.11.1. Case $\rho < 0$, $\lambda = 1$, $\epsilon = -1$. The metric

$$(3.50) \quad h(t) = \frac{\sqrt{|\rho|} \sin(2\sqrt{|\rho|}(t - t_0))}{2 \left(\cos(2\sqrt{|\rho|}(t - t_0)) - \cos \sqrt{|\rho|}s \right)}$$

can be taken to be defined for $t \in (t_0, t_0 + \pi/(2\sqrt{|\rho|}))$ in the bounded open cylinder $s \in (2(t - t_0), 2\pi/\sqrt{|\rho|} - 2(t - t_0))$, or for $t \in (t_0 + \pi/(2\sqrt{|\rho|}), t_0 + \pi/(\sqrt{|\rho|}))$ in the bounded open cylinder $s \in (2\pi/\sqrt{|\rho|} - 2(t - t_0), 2(t - t_0))$. However, a straightforward calculation shows $h(\pi/\sqrt{|\rho|} - s, -t + 2t_0 + \pi/(2\sqrt{|\rho|})) = h(s, t)$, so that, modulo an orientation-reversing unimodular transformation of the time parameter, these flows are equivalent modulo a reflection in s . For this reason, $h(t)$ will be considered for $t \in (t_0, t_0 + \pi/(2\sqrt{|\rho|}))$.

The curvature

$$(3.51) \quad \mathcal{R}_{h(t)} = \frac{2\sqrt{|\rho|}}{\sin(2\sqrt{|\rho|}(t - t_0))} \frac{\cos(2\sqrt{|\rho|}(t - t_0)) \cos \sqrt{|\rho|}s - 1}{\cos(2\sqrt{|\rho|}(t - t_0)) - \cos \sqrt{|\rho|}s},$$

of $h(t)$ is always negative. It is unbounded from below, blowing up as $s \rightarrow 2(t - t_0)$ or $s \rightarrow 2\pi/\sqrt{|\rho|} - 2(t - t_0)$. Since $\mathcal{R}_{h(t)}$ is negative, it follows from $\mathcal{R}_{h(t)}^2 = \mathcal{F}_{h(t)}^2 + \sigma$ that $\mathcal{R}_{h(t)}$ assumes its maximum at a point at which there vanishes $\mathcal{F}_{h(t)}$, should there be such a point in the domain of $h(t)$. Whatever is t_0 , there is always some odd integer k such that the equatorial circle $s = k\pi/\sqrt{|\rho|}$ is contained in the domain of $h(t)$, and along this circle there holds $\mathcal{F}_{h(t)} = 0$, so along this circle \mathcal{R}_h assumes its maximum, which is $-\sqrt{\sigma} = -2\sqrt{|\rho|} \csc(2\sqrt{|\rho|}(t - t_0))$.

3.11.2. Case $\rho < 0$, $\lambda = i$, $\epsilon = -1$. The metric

$$(3.52) \quad h(t) = \frac{\sqrt{|\rho|} \sinh(2\sqrt{|\rho|}(t - t_0))(dr^2 + ds^2)}{2 \left(\cosh(2\sqrt{|\rho|}(t - t_0)) - \cos \sqrt{|\rho|}s \right)}.$$

is defined for all $t > t_0$ and all $s \in \mathbb{R}$, that is on the infinite cylinder. Because $h(t)$ has period $2\pi/\sqrt{|\rho|}$ in s , it descends to the torus $\{(r, s) \in [0, 2\pi) \times [0, 2\pi/\sqrt{|\rho|})\}$.

The curvature

$$(3.53) \quad \mathcal{R}_{h(t)} = \frac{2\sqrt{|\rho|}}{\sinh(2\sqrt{|\rho|}(t - t_0))} \frac{\cosh(2\sqrt{|\rho|}(t - t_0)) \cos \sqrt{|\rho|}s - 1}{\cosh(2\sqrt{|\rho|}(t - t_0)) - \cos \sqrt{|\rho|}s},$$

assumes both positive and negative values. Since $\sigma = \mathcal{R}_{h(t)}^2 + \mathcal{F}_{h(t)}^2 \geq \mathcal{R}_{h(t)}^2$ there holds

$$(3.54) \quad -\sqrt{\sigma} \leq \mathcal{R}_{h(t)} \leq \sqrt{\sigma} = 2\sqrt{|\rho|} \operatorname{csch}(2\sqrt{|\rho|}(t - t_0)),$$

and the maximum and minimum are attained, when $s = 0$ and $s = \pi/\sqrt{|\rho|}$, respectively.

This Ricci flow is interesting because it exists for all $t > t_0$, the manifold is compact, and the $h(t)$ have bounded curvature.

As $t \rightarrow t_0$, the family of homothetic metrics

$$(3.55) \quad k(t) = \sqrt{\sigma}h(t) = \frac{|\rho|(dr^2 + ds^2)}{\cosh(2\sqrt{|\rho|}(t - t_0)) - \cos \sqrt{|\rho|}s},$$

tends pointwise to the the constant curvature -1 hyperbolic metric $|\rho|(1 - \cos \sqrt{|\rho|}s)^{-1}(dr^2 + ds^2)$ on the infinite cylinder.

3.11.3. In all the cases with $\sigma > 0$ and $\rho < 0$ the metrics obtained for different values of ρ are equivalent by a rescaling in s ; on the other hand such a rescaling implicitly rescales Y , so with Y fixed, the absolute value of the parameter ρ remains meaningful from the point of view of consideration of solutions of (1.1).

3.12. **Cases with $\sigma < 0$.** If $\sigma < 0$ then by (3.20) it must be that $\rho > 0$ and $b^2 > a^2$ and so w has the form $w = a \cosh \sqrt{\rho}s + b \sinh \sqrt{\rho}s$. As s may be replaced by $-s$ with no loss of generality, it may be supposed that $b > 0$. In this case there is a unique $q \in \mathbb{R}$ such that $a \cosh \sqrt{\rho}s + b \sinh \sqrt{\rho}s = |\sigma|^{1/2} \sinh(\sqrt{\rho}s + q)$. Since an equivalent geometric structure is obtained upon translation of s , in this case it can be supposed that w has the form

$$(3.56) \quad w = \rho^{-1}(\tau + \sqrt{|\sigma|} \sinh \sqrt{\rho}s).$$

If w is as in (3.56) and $\tau(t)$ is as in (3.23) then the resulting pair $(h(t), Y)$ solves (1.2) while the metrics $h(t)$ constitute a Ricci flow. Explicitly, for $t \in \mathbb{R}$, the metric $h(t)$ is defined on the open half cylinder $s > 2(t - t_0)$ by

$$(3.57) \quad \begin{aligned} h(t) &= \frac{\sqrt{\rho} \cosh(2\sqrt{\rho}(t_0 - t))(dr^2 + ds^2)}{2(\sinh(2\sqrt{\rho}(t_0 - t)) + \sinh \sqrt{\rho}s)} \\ &= \frac{\sqrt{\rho} \cosh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}-2}|dz|^2}{(|z|^{2\sqrt{\rho}} + 2\sinh(2\sqrt{\rho}(t_0 - t))|z|^{\sqrt{\rho}} - 1)} = \frac{\sqrt{\rho} \cosh(2\sqrt{\rho}(t_0 - t))|\tilde{z}|^{\sqrt{\rho}-2}|d\tilde{z}|^2}{(1 + 2\sinh(2\sqrt{\rho}(t_0 - t))|\tilde{z}|^{\sqrt{\rho}} - |\tilde{z}|^{2\sqrt{\rho}})}, \end{aligned}$$

where $\tilde{z} = z^{-1}$. From the last expression in (3.57) it is apparent that the metric $h(t)$ has a conical singularity at the point at infinity with angle $\pi\sqrt{\rho}$. In particular, in the case $\rho = 4$, the metric $h(t)$ extends smoothly to the point at infinity.

The Ricci flow $h(t)$ is remarkable because it is eternal (exists for all time). This does not contradict the main theorem of [6], which classifies complete eternal Ricci flows on surfaces having bounded curvature and bounded width, because the scalar curvature of the metric $h(t)$ of (3.57) is

$$(3.58) \quad \mathcal{R}_{h(t)} = \frac{2\sqrt{\rho}}{\cosh(2\sqrt{\rho}(t_0 - t))} \frac{\sinh(2\sqrt{\rho}(t_0 - t)) \sinh \sqrt{\rho}s - 1}{\sinh(2\sqrt{\rho}(t_0 - t)) + \sinh \sqrt{\rho}s},$$

and as $s \rightarrow 2(t - t_0)$ the curvature tends to $-\infty$.

There holds

$$(3.59) \quad \mathcal{F}_{h(t)} = \frac{2\sqrt{\rho} \cosh \sqrt{\rho}s}{\sinh(2\sqrt{\rho}(t_0 - t)) + \sinh \sqrt{\rho}s}.$$

Observe that

$$(3.60) \quad \mathcal{R}_h + \mathcal{F}_h = \frac{2\sqrt{\rho} (\cosh(\sqrt{\rho}(s + 2(t_0 - t))) - 1)}{\cosh(2\sqrt{\rho}(t_0 - t)) (\sinh(2\sqrt{\rho}(t_0 - t)) + \sinh \sqrt{\rho}s)}$$

is always positive. On the other hand, since $\sigma < 0$ it must be $\mathcal{R}_h - \mathcal{F}_h < 0$. Recall that Lemma 2.2 implies that $\mathcal{R}_h + \mathcal{F}_h$ and $\mathcal{R}_h - \mathcal{F}_h$ must have definite signs. That here $\mathcal{R}_h + \mathcal{F}_h$ is always positive results from the normalizations of s in the construction of $h(t)$, which could be recast invariantly as demanding that Y be such that the sign of $\mathcal{R}_h + \mathcal{F}_h$ is positive.

3.13. Constant curvature metrics yield trivial solutions of (1.1). These can be obtained as limits of certain rescalings of the metrics in (3.28) via the identity

$$(3.61) \quad \lim_{\rho \rightarrow 0} \frac{4}{\rho} u \left(\frac{2s}{\sqrt{|\rho|}}, t \right) = \frac{4(t_0 - t)}{1 + \epsilon \cosh(2\sqrt{\text{sgn}(\rho)}s)} = \begin{cases} \frac{2(t_0 - t)}{\cosh^2(\sqrt{\text{sgn}(\rho)}s)} & \text{if } \epsilon = 1, \\ \frac{2(t - t_0)}{\sinh^2(\sqrt{\text{sgn}(\rho)}s)} & \text{if } \epsilon = -1. \end{cases}$$

The conformal factors u of the limiting metrics, and the corresponding parameters, are

$$(3.62) \quad \begin{array}{lll} \rho > 0, \epsilon = 1 & \rho > 0, \epsilon = -1 & \rho < 0, \epsilon = 1 \\ 2(t_0 - t) \text{sech}^2 s & 2(t - t_0) \text{csch}^2 s & 2(t - t_0) \sec^2 s. \end{array}$$

These are simply constant curvature metrics flowing by homotheties. In the case $\rho > 0$ and $\epsilon = 1$, for all $t < t_0$, the metric $h(t)$ extends to the round metric on the sphere of constant positive scalar curvature $1/(t_0 - t)$. In the case $\rho > 0$ and $\epsilon = -1$, the function u is positive on the positive half-plane $\mathbb{H}_+ = \{s > 0\}$ for all $t > t_0$; the resulting metric $h(t)$ on \mathbb{H}_+ has constant negative scalar curvature $1/(t - t_0)$. In the case $\rho < 0$ and $\epsilon = 1$, the function u is positive if $s \in (0, \pi)$ for all $t > t_0$; the resulting metric $h(t)$ on the strip $\mathbb{R} \times (0, \pi)$ has constant negative scalar curvature $1/(t - t_0)$.

3.14. It has been shown that if (h, Y) solves the real vortex equations (1.1) then there is a solution $h(t)$ of the Ricci flow for which $(h(t), Y)$ solves the real vortex equations (1.1) with a vortex parameter depending on t . This suggest that the Ricci flow can be interpreted as a flow obtained by varying the vortex parameter. It would be interesting to describe this phenomenon precisely and to explain it conceptually. That something along these lines should be the case for one-vortices was proposed by N. Manton in [15], although the relation, if any, between Manton's proposal and the observations described here is not clear.

3.15. **2-vortex solutions on the sphere.** In this section, if \mathbb{W} is a complex vector space, $\mathbb{P}(\mathbb{W})$ denotes its complex projectivization, and the image of $u \in \mathbb{W}$ in the complex projectivization $\mathbb{P}(\mathbb{W})$ is written $[u]$.

3.15.1. Let \mathbb{V} be a two-dimensional complex vector space. Equip \mathbb{V} with a nondegenerate skew-symmetric complex bilinear form $|u, v|$. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is regarded as the Lie algebra of endomorphisms of \mathbb{V} preserving $|u, v|$ infinitesimally in the sense that $|Au, v| + |u, Av| = 0$. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate complex bilinear symmetric form, and let $J \in \text{End}(\mathbb{V})$ be the almost complex structure defined by $|u, v| = \langle Ju, v \rangle$. For $A \in \text{End}(\mathbb{V})$ let A^t be the adjoint endomorphism defined by $\langle Au, v \rangle = \langle u, A^t v \rangle$. Then $A \in \mathfrak{sl}(2, \mathbb{C})$ if and only if $AJ = -JA^t = (AJ)^t$. To an element $A \in \text{End}(\mathbb{V})$ associate the bilinear form $A^\flat(u, v) = \langle Au, v \rangle$, and to a bilinear form $Q \in \mathbb{V} \otimes \mathbb{V}$ associate the endomorphism Q^\sharp defined by $Q(u, v) = \langle Q^\sharp u, v \rangle$. Since Q is symmetric if and only if Q^\sharp is self-adjoint in the sense that $Q^{\sharp t} = Q^\sharp$, the map $\Theta : \mathfrak{sl}(2, \mathbb{C}) \rightarrow S^2(\mathbb{V}^*)$ defined by $\Theta(A) = (JA)^\flat$ (equivalently $\Theta(A)^\sharp = JA$) is a linear isomorphism. Precisely, $\Theta(A)(u, v) = \langle JAu, v \rangle = |Au, v|$.

Let $GL(2, \mathbb{C})$ act on $S^2(\mathbb{V})$ by $(g \cdot \sigma)(u, v) = \sigma(g^{-1}u, g^{-1}v)$ for $g \in GL(2, \mathbb{C})$, $\sigma \in S^2(\mathbb{V}^*)$, and $u, v \in \mathbb{V}$, and on $\text{End}(\mathbb{V}) = \mathfrak{gl}(2, \mathbb{C})$ via the adjoint action. Using that $gJg^t = \det(g)J$ for all $g \in GL(2, \mathbb{C})$ it is straightforward to check that $\Theta(gAg^{-1}) = \det(g)g \cdot \Theta(A)$ for $g \in GL(2, \mathbb{C})$, so that Θ intertwines the restriction to $SL(2, \mathbb{C})$ of these actions. In particular, Θ descends to an isomorphism, also denoted by Θ , between $\mathbb{P}(\mathfrak{sl}(2, \mathbb{C}))$ and $\mathbb{P}(S^2(\mathbb{V}^*))$ that is equivariant with respect to the induced actions of $SL(2, \mathbb{C})$.

An element $A \in S^2(\mathbb{V}^*)$ determines the quadratic form $Q_A(z) = A(z, z)$. For $u, v \in \mathbb{V}$ let $u \odot v = \frac{1}{2}(u \otimes v + v \otimes u)$. Given $u, v \in \mathbb{V}$ the quadratic form

$$(3.63) \quad Z_{u,v}(z, z) = |u, z||v, z| = \langle Ju, z \rangle \langle Jv, z \rangle = Q_{(Ju)^\flat \odot (Jv)^\flat}(z),$$

vanishes exactly at u and v .

Let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V})$ by interchanging the two factors. Since the map $\Psi : \mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V}) \rightarrow \mathbb{P}(S^2(\mathbb{V}^*))$ defined by $\Psi([u], [v]) = [Z_{u,v}]$ is injective on the complement $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \Delta$ of the diagonal Δ , is 2-1 on the diagonal Δ , and is invariant with respect to the action of $\mathbb{Z}/2\mathbb{Z}$ in the sense that $\Psi([v], [u]) = \Psi([u], [v])$, it descends to an identification of the symmetric product $\text{Sym}^2(\mathbb{P}(\mathbb{V})) = (\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V})) / (\mathbb{Z}/2\mathbb{Z})$ with $\mathbb{P}(S^2(\mathbb{V}^*))$.

Via Ψ the action of $SL(2, \mathbb{C})$ on $\mathbb{P}(\mathbb{V})$ by linear fractional transformations induces an action of $SL(2, \mathbb{C})$ on $\mathbb{P}(S^2(\mathbb{V}^*))$. This action has two orbits, the open dense orbit $\Psi(\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V}) \setminus \Delta)$ equal to $SL(2, \mathbb{C})/\mathbb{C}^*$, where the group \mathbb{C}^* of nonzero complex numbers is embedded in $SL(2, \mathbb{C})$ as the diagonal endomorphisms, and the closed complex codimension one orbit $\Psi(\Delta)$ which is the quadric in $\mathbb{P}(S^2(\mathbb{V}^*))$ comprising $[\sigma] \in \mathbb{P}(S^2(\mathbb{V}^*))$ such that $\det \sigma^\sharp = 0$.

Since $\text{tr}((Ju)^\flat \otimes v + (Jv)^\flat \otimes u) = 0$ it makes sense to write

$$(3.64) \quad \Theta\left(\frac{1}{2}((Ju)^\flat \otimes v + (Jv)^\flat \otimes u)\right) = (Ju)^\flat \odot (Jv)^\flat = Z_{u,v}.$$

If $A \in \mathfrak{sl}(2, \mathbb{C})$ has $\det A \neq 0$ then it has two linearly independent eigenvectors u and v in \mathbb{V} . Since $|u, v| \neq 0$, by rescaling u and v appropriately it may be assumed that $2Au = -|u, v|u$ and $2Av = |u, v|v$. Then $A = \frac{1}{2}((Ju)^\flat \otimes v + (Jv)^\flat \otimes u)$, so that $\Theta(A) = Z_{u,v}$. If $0 \neq A \in \mathfrak{sl}(2, \mathbb{C})$ has $\det A = 0$ then there may be chosen u spanning the null space of A such that $AJu = u$, and so $A = (Ju)^\flat \otimes u$ and $\Theta(A) = Z_{u,u}$.

It follows that the composition

$$(3.65) \quad \mathbb{P}(\mathfrak{sl}(2, \mathbb{C})) \xrightarrow{\Theta} \mathbb{P}(S^2(\mathbb{V}^*)) \xrightarrow{\Psi^{-1}} \text{Sym}^2(\mathbb{P}(\mathbb{V})),$$

is an isomorphism, associating to $[A] \in \mathbb{P}(\mathfrak{sl}(2, \mathbb{C}))$ the pair $[[u], [v]] \in \text{Sym}^2(\mathbb{P}(\mathbb{V}))$ comprising its eigenspaces; in particular the quadric $\{[A] \in \mathbb{P}(\mathfrak{sl}(2, \mathbb{C})) : \det A = 0\}$ is mapped isomorphically onto $\Psi(\Delta)$.

To an element $A \in \mathfrak{sl}(2, \mathbb{C})$ associate the holomorphic vector field X^A on $\mathbb{S}^2 = \mathbb{P}(\mathbb{V})$ defined by $X_p^A = \frac{d}{dt} \exp(tA)p$. For $A \neq 0$, the vector field X^A has a single zero of multiplicity 1 or two distinct zeros as $\det A$ is or is not null. In either case the zeros of X^A are the images in $\mathbb{P}(\mathbb{V})$ of the eigenspaces of A . The vector fields X^{A_1} and X^{A_2} have the same zeros if and only if there is $\lambda \in \mathbb{C}^*$ such that $A_2 = \lambda A_1$, that is if and only if their images $[A_1]$ and $[A_2]$ in $\mathbb{P}(\mathfrak{sl}(2, \mathbb{C}))$ coincide. It follows that gauge equivalence classes of holomorphic vector fields on $\mathbb{P}(\mathbb{V})$ are parameterized by $\mathbb{P}(\mathfrak{sl}(2, \mathbb{C}))$ in such a way that the point of $\mathbb{P}(\mathfrak{sl}(2, \mathbb{C}))$ corresponding to a given holomorphic vector field is the point corresponding via $\Theta^{-1} \circ \Psi$ to the zeros of the vector field.

3.15.2. Since $\mathbb{P}^2(\mathbb{C})$ is the moduli space of gauge equivalence classes of 2-vortex solutions of the Abelian Higgs equations, the preceeding suggests that it should be possible to associate such a solution to each gauge equivalence class of holomorphic vector fields.

Fix a reference vector $e \in \mathbb{V}$. Choose coordinates z_1 and z_2 on \mathbb{V} so that $z_1 e + z_2 J e$ is the general element of \mathbb{V} and let $z = z_1/z_2$ be the corresponding inhomogeneous coordinate on $\mathbb{P}(\mathbb{V})$. The gauge equivalence class of holomorphic vector fields vanishing at $[e]$ and $[J e]$ is represented by $iz \partial_z = \partial_r^{(1,0)}$, where the conventions regarding the coordinates and the notations are as in sections 3.1 and 3.2. By Lemma 2.1 any other real vector field X such that $X^{(1,0)}$ represents this gauge equivalence class of holomorphic vector fields is a nonzero real multiple of $Y = \partial_r$. Let $h(t)$ be the sausage metric (3.31) and associate to the point $[[e], [J e]] \in \text{Sym}^2(\mathbb{P}(\mathbb{V})) \setminus \Psi(\Delta)$ the gauge equivalence class of pairs $(h(t), Y)$ solving the real vortex equations (and so the corresponding gauge equivalence class of solutions to the Abelian Higgs equations).

Now let $[[u], [v]] \in \text{Sym}^2(\mathbb{P}(\mathbb{V})) \setminus \Psi(\Delta)$. Then there is $g \in SL(2, \mathbb{C})$ such that $[[gu], [gv]] = [[e], [J e]]$. The pullback $(g^* h(t), g^* Y)$, where g acts in the inhomogeneous coordinate z via linear fractional transformations and $g^* Y$ means the pullback of the real vector field Y via the diffeomorphism given

by the action of g , represents the sought after gauge equivalence class of solutions to the real vortex equations associated to $[[u], [v]]$. From (3.31) it follows that $g^*h(t)$ has the explicit form

$$(3.66) \quad g^*h(t) = \frac{2 \sinh(4(t_0 - t)) |dz|^2}{|az+b|^4 + 2 \cosh(4(t_0 - t)) |az+b|^2 |cz+d|^2 + |cz+d|^4},$$

where $gz = (az+b)/(cz+d)$ with respect to the coordinate z . Though this shows how gauge equivalence classes of solutions to the real vortex equations on $\mathbb{P}(\mathbb{V})$ are parameterized by $\text{Sym}^2(\mathbb{P}(\mathbb{V})) \setminus \Psi(\Delta)$, the explicit form (3.66) does not seem particularly useful. Though one knows that there is a gauge equivalence class of solutions to the Abelian Higgs equations corresponding to an element of the diagonal $\Psi(\Delta)$, this equivalence class admits no representative solving the real vortex equations, for the $(1, 0)$ part of a real Killing field on \mathbb{S}^2 must have two zeros (see Lemma 10.2 of [8]). It is not clear how to write the corresponding metrics explicitly, nor how to realize them as limiting forms of the sausage metrics (3.66), although it seems that this should be possible.

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